A NOTE ON UTILITY INDIFFERENCE PRICING

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Abstract

Utility-based valuation methods are enjoying growing popularity among researchers as a means to overcome the challenges in contingent claim pricing posed by the many sources of market incompleteness. However, we show that under the most common utility functions (including CARA and CRRA), any realistic and actually practicable hedging strategy involving a possible short position has infinitely negative utility. We then demonstrate for utility *indifference prices* (and also for the related so-called *utility-based (marginal) prices*) how this problem leads to a severe divergence between results obtained under the assumption of continuous trading and realistic results. The combination of continuous trading and common utility functions is thus not justified in these methods, raising the question of whether and how results obtained under such assumptions could be put to real-world use.

4.1 Introduction

In recent years, utility indifference pricing, initially proposed by Hodges and Neuberger (1989) and refined by Davis et al. (1993), has gained much attention in the literature on pricing and hedging contingent claims (see Henderson and Hobson (2009) for a survey). Utility indifference pricing employs the tools of continuous time finance, combining the ideas of Black and Scholes (1973) and utility theory. However, as we will demonstrate, the assumptions used by many contributions miss an important step in the link between the idealistic model and reality. Their results cannot be applied practically and it is unclear what insight they could provide for practical problems.

There are several sub-strands of literature which are affected by our findings, namely indifference pricing and hedging in incomplete markets in general, including Barrieu and El Karoui (2009), Biagini and Frittelli (2005), Collin-Dufresne and Hugonnier (2007, 2013), Duffie et al. (1997), Frei and Schweizer (2010), Grasselli and Hurd (2007), Henderson and Hobson (2011), Henderson et al. (2014), Hu et al. (2005), Kramkov and Sîrbu (2006), Malamud et al. (2013), Mania and Schweizer (2005), Musiela and Zariphopoulou (2004a,b), Rheinländer and Steiger (2010), and Svensson and Werner (1993), and the duality methods for the underlying optimization problem, including Becherer (2004), Delbaen et al. (2002), Frittelli (2000a,b), İlhan et al. (2005), İlhan and Sircar (2006), Kabanov and Stricker (2002), Kallsen and Rheinländer (2011), Monoyios (2006), and Rouge and El Karoui (2000). Also contributions on trading restrictions and substitute hedging, like basis risk, basket/index options, and real options, including Becherer (2003), Davis (2006), Frei and Schweizer (2008), Henderson (2002, 2007), Henderson and Hobson (2002a,b), Karatzas and Kou (1996), and Monoyios (2004b) are subject to our results, as well as studies on employee stock options (Grasselli and Henderson 2009; Henderson 2005; Leung and Sircar 2009a,b; Rogers and Scheinkman 2007) and on transaction costs in derivatives pricing (Barles and Soner 1998; Constantinides and Zariphopoulou 1999; Davis et al. 1993; Davis and Yoshikawa 2015; Davis and Zariphopoulou 1995; Hodges and Neuberger 1989; Mohamed 1994; Monoyios 2003, 2004a).

Inspired by indifference pricing, and likewise affected, are contributions on so-called *utility-based prices*, also known as *neutral or shadow prices*, and their marginal version. Contributions not already mentioned above include Hugonnier et al. (2005), Kallsen and Kühn (2004, 2006), Kramkov and Hugonnier (2004), and Owen and Žitković (2009).

The assumption of continuous trading allowed Black and Scholes (1973) to derive a unique option price based on arbitrage arguments that make no assumptions about the market participants' preferences. This ground-breaking idea was developed further towards the concept of a complete market, in which any derivative can be perfectly replicated by continuously trading in the underlyings and thus uniquely priced by arbitrage arguments (see e.g. Delbaen and Schachermayer 1994).

Of course, in practice the time between two hedges is finite. Let us assume that for a *practicable trading strategy*, this time cannot be shorter than some $\delta > 0$. The idealistic assumption of continuous trading is justified by the fact that a continuous strategy H can be approximated to arbitrary accuracy through practicable strategies H_δ with smaller and smaller δ. The proceeds of the self-financing strategy H in a market with discounted price process S are given by the stochastic integral $H \cdot S$, whose mere definition (see e.g. Bichteler 2002) guarantees the existence of H_δ with $H_\delta \cdot S \to H \cdot S$ as δ tends to zero. However, stated differently, for very small values of δ the results do not depend on the particular value of δ and its influence can be neglected. Therefore, working in the limit $\delta \to 0$ is a justified means to achieving clearer and more general results.

Another classical problem in quantitative finance is that of optimal intertemporal portfolio selection. What is optimal is determined by the investor's preferences — even in complete markets — and is usually modelled using von Neumann–Morgenstern utility. Following the same rationale as above, the assumption of continuous trading is still very appealing and thus in widespread use in the field initiated by Merton (1969, 1971)[1], who pioneered in solving the problem for continuous consumption and trading.

These two areas of research have long been considered as being quite distinct from each other. However, in reality markets are not complete and the unique price broadens into an entire range of arbitrage-free prices. It has been shown in many cases, Davis and Clark (1994) and Soner et al. (1995) for proportional transaction costs and Cvitanić et al. (1999) for unbounded stochastic volatility that this range includes the trivial buy-and-hold strategy that dominates the claim. These examples show that arbitrage-free pricing fails to explain the substantially lower prices observed in options markets and a different pricing methodology is needed.

Acknowledging that pricing of unhedgeable risk has to take the agent's preferences into account, Hodges and Neuberger (1989) proposed combining the two above methods. After Davis et al. (1993) gave a rigorous treatment of their idea, it slowly gathered traction and matured into what is known today as *(utility) indifference pricing and hedging*. The agent following a hedging strategy H assigns the expected utility

$$U(X; H) := \mathbb{E}[u(H \cdot S + X)]$$

to every final wealth X, where u is her utility function. Being able to choose hedging strategies from a given set of admissible strategies $\mathcal{H}$ the highest achievable utility is:

$$\overline{U}(X; \mathcal{H}) := \sup_{H \in \mathcal{H}} U(X; H).$$

Most of our results do not depend on a specific definition of $\mathcal{H}$. We will indicate whenever a specific definition is required.

[1] See Merton (1973a) and Sethi and Taksar (1988) for errata.

The *indifference buy price* $p = p(X, W; \mathcal{H}) \in \mathbb{R}$ for a claim X is defined such that the hedging agent with initial random endowment W is indifferent between doing nothing and buying the claim for that price:

$$\overline{U}(W + X - p; \mathcal{H}) = \overline{U}(W; \mathcal{H}). \tag{4.1}$$

A number $b \in \mathbb{R}$ is in the set of *utility-based prices*, $B(X, W; \mathcal{H})$, if it is optimal not to trade the claim at price b:

$$\overline{U}(W; \mathcal{H}) \geq \overline{U}(W + q(X - b); \mathcal{H}), \quad \text{for all } q \in \mathbb{R}. \tag{4.2}$$

One typically looks at $B(X, x + qX; \mathcal{H})$ with initial wealth $x \in \mathbb{R}$ and initial quantity $q \in \mathbb{R}$ in the claim X. Elements of $B(X, x; \mathcal{H})$ are called *marginal prices* of X.

The contribution of this work is to raise and answer the question of whether the assumption of continuous trading is still justified in this setting. As we will argue in the next sections, the answer is *no* if u is exponential or u' approaches ∞ at some finite wealth. This concerns, for example, large part of HARA utility functions (including the logarithmic limiting case) and thus a vast amount of published literature on the topic. The results raise doubts about the use of such utility functions for indifference pricing and related fields in general.

In section 4.2 we show that the assumption fails in cases where the optimal strategy includes shorting one or more assets. Interestingly, this is also what differentiates Merton's portfolio problem from indifference pricing: when solving the problem of "lifetime portfolio selection", negative excess returns or negative equity risk premiums for risky assets would be considered implausible. Thus, the optimal strategy is *long*. However, in indifference pricing the optimal hedging strategy against a shorted put is *short*—in accordance with intuition and other theories. Even though, both areas exhibit this problem on the formal level, it only negatively impacts the practical relevance of the more recent theory of indifference pricing.

Sections 4.3 and 4.4 use the example of a dynamically replicable claim to demonstrate how indifference pricing and utility-based pricing, respectively, are affected by the results in section 4.2. Section 4.5 discusses some failed resolutions attempts and gives a first characterization of utility functions not suffering from these difficulties. We conclude with section 4.6.

4.2 Failure of the continuous trading assumption

The discounted price process of the risky asset is given by S, a real valued semimartingale on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$. A trading strategy H is an element of $L(S)$, the set of predictable S-integrable processes. The self-financing result of H is given by the stochastic integral $H \cdot S$. Payoffs of claims are modeled by $\mathcal{F}$-measurable random variables. Comparison operators on random variables are understood in the $\mathbb{P}$-almost sure sense. We will need the set of *practicable* hedging strategies $\mathcal{P}$, where rebalancing happens at a finite number of fixed times and positions have to be closed if the wealth at the time of trade drops below a certain negative threshold.

Definition 4.1 (Practicable hedging strategies). *A strategy $H \in L(S)$ is in $\mathcal{P}$, if there exists an increasing finite sequence of fixed times $\{t_i\} \subset [0, \infty)$ and an $M \in \mathbb{R}$, such that*

$$H = \sum_i h_i \mathbb{1}_{(t_i, t_{i+1}]},$$

where for any i, h_i is an $\mathcal{F}_{t_i}$-measurable random variable, and

$$H \cdot S^{t_i} \leq -M \Rightarrow h_i = 0 \tag{4.3}$$

holds almost surely. ($S_t^{t_i} := S_{\min(t_i, t)}$ is the price process stopped at t_i.)

By keeping in mind that the above definition includes strategies of several billion possible trades per second, it is obvious that our definition of *practicable strategies* is in no way a restriction of what is actually possible in practice.

It is also this concept of practicable strategies that is used in academic publications and text books to motivate the definition and use of continuous-time concepts.

Remark 4.1. We leave it to future research to extend the following arguments to handle a more general definition of $\mathcal{P}$, where trades happen at stopping times and the minimal time between two trades is finite.

The restriction in eq. (4.3) was introduced to exclude so-called doubling or martingale strategies. It is far less restrictive than margin requirements found in practice which are constantly monitored and enforced and usually limit the position size to a multiple of the available collateral.

Remark 4.2. Pricing theories, where continuous trading is allowed, exclude such unwanted strategies by the definition of $\mathcal{H}$, which usually requires $H \cdot S^t$ to be bounded from below for all t. For practicable strategies, however, this is too restrictive, as it would disallow any short position. Only the (unrealistic) assumption of continuous trading makes it possible to limit the unbounded risk of a short position.

Furthermore, we define the set $\mathcal{S}$ of practicable hedging strategies that include a possible *short position*.

Definition 4.2. $\mathcal{S} = \{H \in \mathcal{P} \mid \exists\, t : \mathbb{P}(H_t < 0) > 0\}$

For the rest of this section, we make the following assumption about the utility function and market model which covers many common settings and is enough to prove the failure without much effort.

Assumption 4.1. *We assume the utility function, $u : \mathbb{R} \to \overline{\mathbb{R}}$, is a concave and non-decreasing function. Furthermore, we assume one of the following cases:*

Case 1. *(a) $u(x) = -e^{-\gamma x}$ for some $\gamma > 0$ and*
(b) the distribution of the stock price has heavy right tails, i.e.

$$\lim_{x\to\infty} e^{\lambda x}\mathbb{P}(S_t > x|\mathcal{F}_s) = \infty, \textit{for any } \lambda > 0 \textit{ and } t > s.$$

Case 2. *(a) u satisfies the Inada condition*

$$\lim_{x\downarrow B} u'(x) = \infty \textit{ for some } B \in \mathbb{R}$$

and (b) $S_t - S_s$ is unbounded for any $t > s > 0$.

Remark 4.3 (Case 1)*.* Condition (b) is even satisfied for log-normally distributed S_t. As a consequence, practically all continuous-time models and models for stock returns used in literature possess heavy tails in the sense of Case 1(b).

Empirical evidence suggests (Ibragimov et al. 2015) that the distributions of the *logarithms* of returns are heavy tailed, a requirement going much further than the above condition.

Remark 4.4 (Case 2)*.* Utility functions covered by Case 2(a) include, among others, all utility functions with hyperbolic absolute risk aversion (HARA) and exponent $\gamma \in (-\infty, 1)$:

$$u(x) = \frac{1}{\gamma}((x-B)^\gamma - 1). \tag{4.4}$$

In the limit $\gamma \to 0$, this also includes the logarithmic utility function $u(x) = \ln(x - B)$. Setting $B = 0$ yields utility functions with constant relative risk aversion (CRRA), also called isoelastic utility functions.

All of these functions have domains $D \subseteq [B, \infty)$. We will use their *unique* concave extensions given by $u = -\infty$ on $\mathbb{R}\backslash D$. Without extension, the original utility functions could not be used at all due to the possibility of unbounded losses incurred by short positions in the underlying. Of course, such an extension does not alter any pre-existing results.

The core of the problem is that under Assumption 4.1, any practicable strategy with possible short positions has infinitely negative utility.

Theorem 4.1. *Consider a payoff X, bounded from above, and a strategy $H \in \mathcal{S}$. If $U(X; H) = \mathbb{E}[u(H\cdot S + X)]$ exists, then its value is $-\infty$.*

Proof. The expectation of a random variable Y taking values in $\overline{\mathbb{R}}$ is defined (e.g. by Doob 1994, VI.4.) as $\mathbb{E}[Y^+]-\mathbb{E}[Y^-]$, if one of these non-negative expectations is finite, where $Y^\pm := \max(\pm Y, 0)$. Thus, it suffices to show $\mathbb{E}[u(H\cdot S + X)^-] = \infty$.

Without loss of generality, we assume the strategy is short on $(0, \tau]$, with $H_\tau = -h$ for some real $h > 0$. Other cases can be handled by taking expectations conditioned on the instant of time of opening the first short position.

Now take an arbitrary real number $x > M/h + S_0$ with M from Definition 4.1 and prove:

$$\begin{aligned}\mathbb{E}[u(H\cdot S + X)^-] &\geq \mathbb{E}\left[\mathbb{1}_{S_\tau > x}\, u(h(S_0 - S_\tau) + X)^-\right] \\ &\geq \mathbb{E}\left[\mathbb{1}_{S_\tau > x}\, u(h(S_0 - x) + \sup X)^-\right] \\ &= u\big(h(S_0 - x) + \sup X\big)^- \mathbb{P}(S_\tau > x). \end{aligned} \tag{4.5}$$

The first inequality uses the fact that $Y \geq \mathbb{1}_{S_\tau > x}\, Y$ for any positive Y. Furthermore, it applies the restriction from eq. (4.3): If $S_\tau > x$, then $H\cdot S^\tau = h(S_0 - S_\tau) < -M$ and thus all hedging positions after τ are zero. The second inequality uses the monotonicity of u^- together with $X \leq \sup X$ and $h(S_0 - S_\tau) < h(S_0 - x)$, if $S_\tau > x$.

Taking the limit $x \to \infty$ of eq. (4.5) in *Case 1* yields due to $\sup X < \infty$:

$$\mathbb{E}[u(H\cdot S + X)^-] \geq e^{-\gamma(hS_0 + \sup X)} \lim_{x\to\infty} e^{\gamma h x}\mathbb{P}(S_\tau > x) = \infty.$$

In *Case 2* due to $\sup X < \infty$, we can choose an x such that $h(S_0 - x) + \sup X < B$ and thus $u(h(S_0 - x) + \sup X)^- = \infty$ by u's concavity (see also Remark 4.4). The unboundedness of S_τ implies $\mathbb{P}(S_\tau > x) > 0$ and consequently eq. (4.5) proves $\mathbb{E}[u(H\cdot S + X)^-] = \infty$. □

Remark 4.5. Theorem 4.1 considers only X that are bounded from above, which includes, for example, the common cases of long or short put options and short call options.

A more general version would complicate the proof. In particular applications, however, the requirement of boundedness can be significantly relaxed without much effort. Theorem 4.1 holds as long as X does not exhibit asymptotic long exposure in the asset S, ensuring that X does not effectively neutralize a short hedging position for large values of S.

An immediate consequence of Theorem 4.1 is that short strategies are practically forbidden for the agent and excluded from the utility optimization over any set $\mathcal{H}$ of hedging strategies, that is

$$\overline{U}(X;\mathcal{H}) = \overline{U}(X;\mathcal{H} \setminus \mathcal{S}). \tag{4.6}$$

Theorem 4.1 contradicts the fact that short positions are used in practice and consequently Assumption 4.1 has to be rejected.

Furthermore, it implies that the assumption of continuous trading is not justified: Consider a continuous strategy $H \in L(S)$ that has a positive probability of a short position. Any sensible practicable approximation $K \in \mathcal{P}$ of this strategy will then be an element of $\mathcal{S}$ and thus have infinitely negative utility.

This affects, for example, the standard approximation for strategies with continuous paths, as used in Monoyios (2004b):

$$K\cdot S = \sum_i H_{t_i}\big(S_{t_{i+1}} - S_{t_i}\big), \text{ for some } (t_i) \in \mathbb{R}^n .$$

In fact, approximating the optimal continuous-time strategy has strictly lower utility than the zero strategy:

$$U(X;0) > U(X;K), \quad (= -\infty, \text{ by Theorem 4.1}).$$

Or stated differently: the agent will always prefer *not to hedge at all.*

In summary, results obtained under Assumption 4.1 and the assumption of continuous trading are disconnected from what is possible in reality.

4.3 Implications for indifference pricing

In this section, we give simple examples that explicitly demonstrate how things can go wrong under Assumption 4.1 and why neither the price nor the hedging strategy derived under the continuous trading assumption have any meaning for an agent limited to practicable strategies.

For demonstration purposes, we price and hedge dynamically replicable contingent claims. It is well known, that under the assumption of continuous trading (with a suitable $\mathcal{H}$) the utility indifference price of a replicable claim equals the arbitrage-free price for a broad range of utilities.[2]

In this section, we will use the optimal strategy with claim, $H^*(X;\mathcal{H})$ and without a claim, $Z(\mathcal{H})$, defined by

$$\overline{U}(0;\mathcal{H}) = U(0;Z(\mathcal{H})) = U(X - p(X,0;\mathcal{H}); H^*(X;\mathcal{H})).$$

4.3.1 *General observations*

The following notation will be employed: the replicable claim's payoff is given by $\Delta \cdot S + q$, where q is the arbitrage-free price and Δ the replicating strategy, which is an element of the set of admissible continuous trading strategies $\mathcal{H} \subseteq L(S)$. We assume that q equals the continuous-trading utility indifference price.

Physical reality and eq. (4.6) limit the set of admissible strategies to a subset of practicable long-only strategies $\mathcal{K} \subseteq \mathcal{H} \cap \mathcal{P} \setminus \mathcal{S}$. Replacing $\mathcal{H}$ by $\mathcal{K}$ in eq. (4.1) will yield practicable optimal strategies and prices that substantially differ from their theoretical counterparts.

The following examples need a simple consequence of Jensen's inequality.

Lemma 4.1. *For any set $\mathcal{A} \subseteq \{H \in L(S) \mid \mathbb{E}[H \cdot S] \le 0\}$ it holds $U(0;0) = \overline{U}(0;\mathcal{A})$.*

Proof. $U(0;0) \le \overline{U}(0;\mathcal{A}) = \sup_{H \in \mathcal{A}} \mathbb{E}[u(H \cdot S)] \le \sup_{H \in \mathcal{A}} u(\mathbb{E}[H \cdot S]) \le u(0) = U(0;0)$. □

In our first example, we assume zero initial wealth, $W_1 := 0$, and a claim $X_1 := \Delta \cdot S + q$ bounded from above, $\Delta \ge 0$ and that the stock has zero excess return. Lemma 4.1 entails $Z(\mathcal{H}) = Z(\mathcal{K}) = 0$ and thus $H^*(X_1;\mathcal{H}) = -\Delta$, which

[2] For an early proof see Davis et al. (1993, Theorem 1), or for a more recent presentation Becherer (2003, eq. (3.8)), who calls this *elementary no-arbitrage consistency*.

is negative and by Theorem 4.1 every practicable approximation will have infinitely negative utility. Therefore, in practice it is favorable not to hedge at all. Furthermore, due to this impossibility to hedge, $p_1 := p(X_1, W_1; \mathcal{H}) = q$ is too high. If u is strictly concave, the agent pays too much and strictly decreases her utility when buying the option. We state this result which follows from eq. (4.6) and a strict version of Jensen's inequality without proof:

$$\overline{U}(X_1 - p_1; \mathcal{K}) = \sup_{H \in \mathcal{K}} \mathbb{E}[u((\Delta + H) \cdot S)] < u(0) = \overline{U}(0; \mathcal{K}). \tag{4.7}$$

However, the continuous-trading buy price is not always too high. Take, for example, the indifference price for closing the current risky position X_1. In this case, we have an initial portfolio $W_2 := X_1 - p_1$ and a payoff $X_2 := -X_1$. The continuous-trading price is of course $p_2 := p(X_2, W_2; \mathcal{H}) = -p_1$. For the realistic agent, this price is too low. By closing the position, she is able to eliminate the unhedgeable risk in her current portfolio. With eq. (4.7), we can see that buying X_2 for p_2 strictly increases the agent's utility:

$$\overline{U}(W_2 + X_2 - p_2; \mathcal{K}) = \overline{U}(0; \mathcal{K}) > \overline{U}(X_1 - p_1; \mathcal{K}) = \overline{U}(W_2; \mathcal{K}).$$

Another example, where buying a claim for the continuous indifference price improves the realistic agent's situation, i.e. where the continuous-trading price is again too low, is that of negative excess returns of the stock.

In this case, the optimal continuous strategy $Z(\mathcal{H})$ is short and the optimal practicable strategy is $Z(\mathcal{K}) = 0$ by Lemma 4.1. Now consider the claim with payoff $X_3 := Z(\mathcal{H}) \cdot S$. Of course, its arbitrage-free price is zero and thus $p(X_3, 0; \mathcal{H}) = 0$. However, buying this claim for 0 establishes the optimal short exposure which was not previously available to the practicable agent and thus strictly increases her utility. See Example 4.2 for concrete results in a log-normal model.

The assumption of continuous trading does not fail in situations in which the optimal continuous strategy on both sides of eq. (4.1) is long. One example is the situation where one put is sold from a portfolio whose optimal hedging position is long and only slightly reduced by the sale of one put. See Example 4.3 for concrete results in a log-normal model.

4.3.2 Concrete examples

For all of the following examples, we assume a risk-free rate of zero ($r = 0$), an agent with exponential utility ($u(x) = -e^{-\gamma x}$, with $\gamma = 0.01$), no initial portfolio ($W = 0$) and that the asset price follows a geometric Brownian motion with return μ:

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right).$$

We use the usual $\mathcal{H} = \{H \in L(S) \mid H \cdot S \text{ bounded from below}\}$ and $\mathcal{K} = \mathcal{H} \cap \mathcal{P} \setminus \mathcal{S}$. This market is arbitrage-free with a unique martingale measure $\mathbb{Q}$ and the corresponding risk-neutral expectation $\mathbb{E}_{\mathbb{Q}}$.

Example 4.1. In the first example, we calculate the sell price of a put on `GOOGL` with payoff $X = (550 - S_T)^+$ at a current stock price of $S_0 = 600$, with $\mu = 0$ and $\sigma\sqrt{T} = 0.25$. Lemma 4.1 implies $Z(\mathcal{H}) = Z(\mathcal{K}) = 0$. The continuous-trading indifference sell price is given by the risk-neutral expectation of the payoff, $-p(-X, 0; \mathcal{H}) = \mathbb{E}_{\mathbb{Q}}[X] \approx 35.61$, and the optimal strategy $-\Delta$ is given by the Black and Scholes (1973) delta for a put, i.e. short. Hence as in eq. (4.7), the optimal practicable strategy is zero and the indifference sell price is:

$$-p(-X, 0; \mathcal{K}) = \frac{1}{\gamma} \ln \frac{\overline{U}(-X; \mathcal{K})}{\overline{U}(0; \mathcal{K})} = \frac{1}{\gamma} \ln \frac{U(-X; 0)}{U(0; 0)} = \frac{1}{\gamma} \ln \mathbb{E}\left[e^{\gamma X}\right] \approx 58.50.$$

Example 4.2. In the second example, we assume that every trading strategy $H \in \mathcal{H}$ is constant after a fixed time T, and take a look at the Hamilton-Jacobi-Bellman equation for $\overline{U}(0; \mathcal{H})$. Davis et al. (1993, eq. (4.30)) derive the optimal strategy:

$$Z(\mathcal{H})_t = \frac{\mu}{\gamma \sigma^2 S_t}. \tag{4.8}$$

This strategy replicates a claim with payoff $X := Z(\mathcal{H}) \cdot S = \frac{\mu}{\gamma}\left(\frac{T}{2} + \frac{1}{\sigma^2} \ln \frac{S_T}{S_0}\right)$ and optimal practicable utility $\overline{U}(X; \mathcal{K}) = U(X; 0)$, which can be seen from:

$$U(X; 0) \leq \overline{U}(X; \mathcal{K}) \leq \overline{U}(X; \mathcal{H}) = \overline{U}(0; \mathcal{H}) = U(0; Z(\mathcal{H})) = U(X; 0).$$

The continuous indifference price of this claim is $p(X, 0; \mathcal{H}) = \mathbb{E}_{\mathbb{Q}}[X] = 0$. However, in the case $\mu < 0$ and thus $Z(\mathcal{H}) < 0$ and by Lemma 4.1 $Z(\mathcal{K}) = 0$, its practicable indifference price is strictly positive:

$$p(X, 0; \mathcal{K}) = \frac{-1}{\gamma} \ln \frac{\overline{U}(X; \mathcal{K})}{\overline{U}(0; \mathcal{K})} = \frac{-1}{\gamma} \ln \frac{U(X; 0)}{U(0; 0)} = \frac{-1}{\gamma} \ln \mathbb{E}\left[e^{-\gamma X}\right] = \frac{\mu^2 T}{2\sigma^2 \gamma} > 0.$$

Example 4.3. This time we assume a positive excess return $\mu > 0$. According to eq. (4.8), the optimal strategy without any payoff is positive. The optimal continuous trading strategy after selling x put options with exercise price K is given by $H := H^*(-x(K - S_T)^+; \mathcal{H}) = Z(\mathcal{H}) - x\delta$, where δ is the Black and Scholes (1973) delta of the put. Simple analysis shows that $H \geq 0$ if and only if $x \leq \frac{\mu}{\gamma \sigma^2 K} =: x^*$. Consequently, only then can both $Z(\mathcal{H})$ and H be approximated by practicable strategies and the results obtained under the assumption of continuous trading carry over to the practicable case.

Assuming $\mu = 5\%$, $\sigma = 25\%$, and $K = 550$, we obtain $x^* \approx 0.15$, i.e. when selling more than that fraction of a put, the continuous trading price will differ from the practicable price.

We conclude with the observation that optimal continuous strategies and the corresponding indifference prices are not relevant for a realistic agent.

4.4 Implications for utility-based pricing

As in the previous section, we will look at dynamically replicable claims. Under the continuous trading assumption (with a suitable $\mathcal{H}$) the utility-based price of a replicable claim X is unique and given by the claim's arbitrage-free price (c.f. Kramkov and Hugonnier 2004):

$$B(X, W; \mathcal{H}) = \{\mathbb{E}_{\mathbb{Q}}[X]\}.$$

Under Assumption 4.1, this theoretical result is violated for a practical hedger, with strategies $\mathcal{K} \subseteq \mathcal{P} \cup \{0\}$, in various settings, two of which we will give here as examples. In the first setting, let us assume that the optimal hedge against a non-negative number of claims with payoff given by X is not to hedge at all, i.e.

$$\overline{U}(x + qX; \mathcal{K}) = U(x + qX; 0), \text{ if } q \geq 0. \tag{4.9}$$

Then, the following Theorem proves that utility-based prices for realistic agents with strictly concave u holding positive quantities of a claim are strictly lower than the claim's physical expectation value and thus for replicable claims strictly lower than the continuous trading result from above, whenever $\mathbb{E}[X] \leq \mathbb{E}_{\mathbb{Q}}[X]$.

Theorem 4.2. *If a claim X satisfies eq. (4.9), then for any real x and $q > 0$ it holds* $\sup B(X, x + qX; \mathcal{K}) \leq c$ *with*

$$c \equiv \frac{1}{q}\left(u^{-1}\left(\mathbb{E}\left[u(x + qX)\right]\right) - x\right) \leq \mathbb{E}[X].$$

If u is strictly concave, the last inequality becomes "<".

Proof. $c \leq \mathbb{E}[X]$ holds due to Jensen's inequality or the corresponding strict version for strictly concave u. Using eq. (4.9) and u's strict monotonicity, we can for any $b > c$ derive a violation of eq. (4.2), which proves that $(c, \infty] \not\subseteq B(X, x + qX; \mathcal{K})$:

$$\overline{U}(x+qX; \mathcal{K}) = \mathbb{E}\left[u(x + qX)\right] = u(x+qc) < u(x+qb) = \overline{U}(x+qX-q(X-b); \mathcal{K}).$$

□

Example 4.4 (Closing a short put position). In the exponential utility setting from Example 4.1 consider an agent holding one shorted put. The marginal price with continuous trading is $B(X, -X; \mathcal{H}) = \{\mathbb{E}_{\mathbb{Q}}[X]\} \approx \{35.61\}$. Yet, using the substitution $X \to -X$, Theorem 4.2 provides a lower bound for $B(X, -X; \mathcal{K})$ of $c \approx 58.50$. The exact value (given without proof) is significantly higher: $B(X, -X; \mathcal{K}) = \{\mathbb{E}[u'(-X)X]/\mathbb{E}[u'(-X)]\} \approx \{87.95\}$.

In the second setting, we assume negative excess returns of all market assets, a certain smoothness of u at the current wealth level and a given set of practicable strategies, $\mathcal{K} \subseteq \mathcal{P} \cup \{0\}$. The following theorem and example show that in this case the marginal price of any bounded (not necessarily replicable) claim is given by its physical expectation, which is again a large deviation from the continuous trading result.

Theorem 4.3. *If* $\mathbb{E}[H \cdot S] \leq 0$ *for all* $H \in \mathcal{K}$, *and* u' *exists at* x, *then* $B(X, x; \mathcal{K}) = \{\mathbb{E}[X]\}$ *for all bounded* X.

Proof. Due to $\mathbb{E}[H \cdot S] \leq 0$, Jensen's inequality and u's monotonicity, we have

$$\overline{U}(x + q(X - b); \mathcal{K}) \leq u(x + q(\mathbb{E}[X] - b)), \quad \text{for all } q \in \mathbb{R},$$

and $\overline{U}(x; \mathcal{K}) = u(x)$. This shows, that $b = \mathbb{E}[X]$ fulfills eq. (4.2). Now define $f(q) \equiv U(x + q(X - b); 0)$. Boundedness of X allows us to interchange limit and expectation:

$$\lim_{q \to 0} \frac{f(q) - f(0)}{q} = \mathbb{E}\left[\frac{d}{dq} u(x + q(X - b))\Big|_{q=0}\right] = u'(x)(\mathbb{E}[X] - b).$$

For every $b \neq \mathbb{E}[X]$ this limit is different from 0. Hence, there is some q, such that $\overline{U}(x + q(X - b); \mathcal{K}) \geq f(q) > f(0) = \overline{U}(x; \mathcal{K})$, proving that b violates eq. (4.2). □

Example 4.5 (Marginal put price under exponential utility). Due to Theorem 4.1, a realistic agent cannot short the stocks and thus cannot profit from falling stock prices. However, she can achieve short exposure through put options, and consequently purchasing them even for considerably more than the arbitrage-free price will improve her situation. In the setting of Example 4.4, but with $\mu T = -0.08$, the marginal price with continuous trading is again $B(X, x; \mathcal{H}) = \{\mathbb{E}_{\mathbb{Q}}[X]\} \approx \{35.61\}$, yet Theorem 4.3 gives $B(X, x; \mathcal{K}) = \{\mathbb{E}[X]\} \approx \{52.99\}$.

In analogy to the last section, we demonstrated utility-based prices obtained under the assumption of that continuous trading are too far off to be relevant for realistic agents.

4.5 Discussion

In this section, we address the question of whether there is an easy solution to the problem and give the answer: No.

The first attempt is to employ an economic argument to make the region in which the net position is never short—as in Example 4.3—large enough to cover the cases of interest. Still, outside this region it will always abruptly fail, even in dynamically complete markets. For a pricing and hedging theory this is not satisfactory.

Another attempt is to limit the unbounded losses of short positions through some stopping mechanism. If a stopping time τ exists such that the stopped process S^τ is bounded, then any practicable strategy in S^τ would have finite utility. How can the agent trade in the S^τ? Of course, it can be replicated using a continuous trading strategy and its arbitrage-free price at time t equals S^τ_t. However, under Assumption 4.1 such a strategy is not practicable, neither for the agent nor for the possible issuer of S^τ. The reason for this is that τ can be shorter than any fixed time which also lies at the heart of the problem discussed in this paper. While such products exist in practice (like guaranteed stops or barrier options with knock-out features), they do not resolve the issue. The only scenario in which the spread offered on S^τ enables the agent to approximate a continuous strategy in S, is one where Assumption 4.1 does not hold for the issuer of S^τ. Thus, nothing is gained and the problem not resolved.

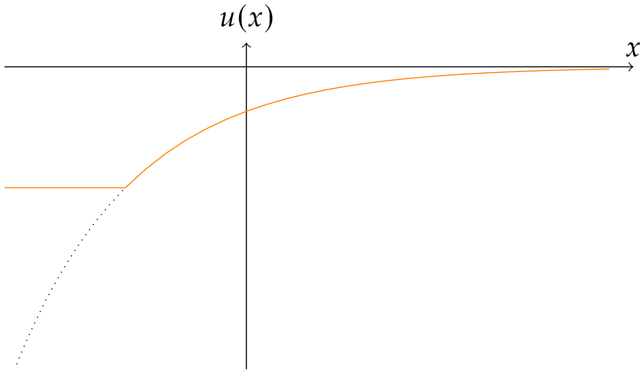

Figure 4.1: Capped exponential utility function.

If limiting the losses is not possible, one alternative could be arguing the fact that an upper bound on the losses always exists, e.g. through limited liability cooperations. Thus, the actor's utility cannot fall below a certain level which effectively corresponds to the situation depicted in Figure 4.1. Such a bound, however, breaks the concavity of the utility function and thus contradicts the risk aversion of the agent, as shown in the following example.

Assume a non-hedging agent is pricing a payoff f with a rare but severe possible loss event, $\mathbb{P}(f = 10.\overline{010}) = 99.9\%$ and $\mathbb{P}(f = -10\,000) = 0.1\%$. Although f has zero expectation, its indifference price under a utility function $u(x) = -e^{-(x^+)}$ is $p = 10.\overline{010}$.

One might ask why the existence of numerical algorithms for continuous trading results does not contradict our findings. After all, most numerical methods like PDE or tree methods somehow discretize the trading strategy (e.g. with mesh size δt) and thus describe a practicable hedging strategy, yet they yield finite results contrary to Theorem 4.1. The reason lies in the additional discretization of the state space (δS) which implies that cases 1(b) and 2(b) from Assumption 4.1 do not apply and the infinities of Theorem 4.1 do not occur. The finite continuum value is then obtained through a simultaneous limit $(\delta S, \delta t) \to 0$. Applying such methods to the practicable case requires keeping δt fixed in the limit $\delta S \to 0$. This limit diverges for short strategies and thus the numerical scheme will again produce the analytically correct, albeit problematic result of $-\infty$.

Last, we would like to address the reports of the seemingly good performance achieved by hedging strategies obtained under Assumption 4.1 (e.g. Mohamed 1994; Monoyios 2004a,b). In these studies, the distribution of the hedging error of different hedging strategies is compared on the basis of statistics such as mean, standard deviation and median. However, first optimizing the utility of the strategy and then looking at another performance measure defeats the purpose. Even if a utility optimizing strategy outperforms others, there will always be a better strategy: the optimal strategy under the measure of interest.

Besides this fundamental problem, the strategies considered in these studies are practicable approximations to the optimal continuous strategy. By Theorem 4.1, such a strategy has infinitely negative utility, if it contains a short position. Still, considering the implementation of such a strategy—e.g. as suggested by the studies above—implies that the agent simply does not have the postulated utility function.

We deem the transition to more suitable functions inevitable. More specifically, a utility function u should not exclude realistic behavior and ideally allow results obtained under the assumption of continuous trading to be applied practically. Obviously, to fulfill these requirements, u must not satisfy Assumption 4.1.

In addition to this necessary condition, we provide a sufficient condition in a simple setting. In accordance with intuition, u has to be defined on $\mathbb{R}$ and should not fall too fast as the wealth approaches negative infinity. Indeed, if S is a square integrable martingale, e.g. a geometric Brownian motion with no drift, it is sufficient for u to have a bounded left derivative. In this case, for any square integrable, replicable claim X there exists a sequence of simple processes $\{q_n\}_n$ that achieve the expected utility of a perfect hedge:

$$U(X; q_n) \xrightarrow[n\to\infty]{} u(\mathbb{E}_{\mathbb{Q}}[X]). \tag{4.10}$$

To see this, we resort to the definition of the stochastic integral, which ensures the existence of a sequence of simple processes $\{q_n\}_n$ such that $Y_n \equiv X - \mathbb{E}_{\mathbb{Q}}[X] + q_n \cdot S$ converges in mean square to zero. Now, if the upper bound of u's left derivative is given by C, then due to u's concavity and monotonicity, we get $|u(a) - u(b)| \leq C|a - b|$ and thus

$$|U(X; q_n) - u(\mathbb{E}_{\mathbb{Q}}[X])| \leq \mathbb{E}\left[|u(X + q_n \cdot S) - u(\mathbb{E}_{\mathbb{Q}}[X])|\right] \leq C\mathbb{E}[|Y_n|] \longrightarrow 0.$$

One utility function with a bounded derivative is $u(x) = kx - \sqrt{1 + (kx)^2}$ for any $k > 0$. The parameter k determines the absolute risk aversion at $x = 0$. This function exhibits a growing relative risk aversion for positive wealth, asymptotically approaches a constant relative risk aversion coefficient with value 2, and thus makes an economically viable candidate.

4.6 Conclusion

While utility indifference pricing and hedging as well as the so-called utility-based pricing are fruitful approaches in conquering the challenges of incomplete markets, we believe this work demonstrates that the use of utility functions that satisfy Assumption 4.1 contradicts even the most basic practical observations. This has demonstrable consequences for the practical applicability of results obtained under the assumptions of continuous trading. Even if mathematically elegant results rely on the simplicity of these utility functions, we plead for a replacement by more realistic ones.

While the last section should serve as a starting point in that matter, the broader question of general practical applicability of indifference and utility-based pricing will intimately depend on the combination of admissible strategies, the utility function, the price dynamics, the payoff and finally the kind of result to be applied and is left to further research.

We conclude with the managerial implication that the practical implementation of hedging strategies derived from continuous-time utility indifference pricing under Assumption 4.1 leads to far from optimal behavior.

BIBLIOGRAPHY

Aber, J. W., D. Li, and L. Can (2009). Price volatility and tracking ability of ETFs. *Journal of Asset Management* **10**(4), 210–221.

Ahn, H., A. Penaud, and P. Wilmott (1999). Various passport options and their valuation. *Applied Mathematical Finance* **6**(4), 275–292.

Ahn, H. and P. Wilmott (2009). A note on hedging: restricted but optimal delta hedging, mean, variance, jumps, stochastic volatility, and costs. *Wilmott Journal* **1**(3), 121–131.

Amihud, Y. and H. Mendelson (1991). Liquidity, asset prices and financial policy. *Financial Analysts Journal* **47**(6), 56–66.

Ammann, M. and J. Tobler (2000). Measurement and decomposition of tracking error variance. *Discussion paper no. 2000-11*.

Anson, M. J. (1999). Maximizing utility with commodity futures diversification. *The Journal of Portfolio Management* **25**(4), 86–94.

Anson, M. J., F. J. Fabozzi, and F. J. Jones (2011). *The Handbook of Traditional and Alternative Investment Vehicles: Investment Characteristics and Strategies*. Hoboken (i.a.): John Wiley & Sons, Inc.

Aroskar, R. and W. A. Ogden (2012). An analysis of exchange traded notes tracking errors with their underlying indexes and indicative values. *Applied Financial Economics* **22**(24), 2047–2062.

Artzner, P., F. Delbaen, J.-M. Eber, and D. Heath (1999). Coherent measures of risk. *Mathematical Finance* **9**(3), 203–228.

Artzner, P., F. Delbaen, J.-M. Eber, D. Heath, and H. Ku (2007). Coherent multiperiod risk adjusted values and Bellman's principle. *Annals of Operations Research* **152**(1), 5–22.

Barles, G. and H. M. Soner (1998). Option pricing with transaction costs and a nonlinear Black-Scholes equation. *Finance and Stochastics* **2**(4), 369–397.

Barrieu, P. and N. El Karoui (2009). Pricing, hedging and optimally designing derivatives via minimization of risk measures. *Indifference pricing: theory and applications*. Ed. by R. Carmona. Princeton University Press, 77–146.

Bayraktar, E., Y.-J. Huang, and Z. Zhou (2015). On hedging american options under model uncertainty. *SIAM Journal on Financial Mathematics* **6**(1), 425–447.

Becherer, D. (2003). Rational hedging and valuation of integrated risks under constant absolute risk aversion. *Insurance: Mathematics and Economics* **33**(1), 1–28.

— (2004). Utility–indifference hedging and valuation via reaction–diffusion systems. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* **460**(2041), 27–51.

Belousova, J. and G. Dorfleitner (2012). On the diversification benefits of commodities from the perspective of euro investors. *Journal of Banking & Finance* **36**(9), 2455–2472.

Bensoussan, A. (1984). On the theory of option pricing. *Acta Applicandae Mathematica* **2**(2), 139–158.

Biagini, S. and M. Frittelli (2005). Utility maximization in incomplete markets for unbounded processes. *Finance and Stochastics* **9**(4), 493–517.

Bichteler, K. (2002). *Stochastic integration with jumps*. Vol. 89. Encyclopedia of Mathematics and its Applications. Cambridge University Press.

Bienkowski, N. (2007). Exchange traded commodities: led by gold, ETCs opened the world of commodities to investors. *Alchemist – The London Bullion Market Association* **48**, 6–8.

— (2010). Oil futures, exchange-traded commodities and the oil futures curve. *Journal of Indexes* **13**(3), 40–43.

Black, F. and M. Scholes (1973). The pricing of options and corporate liabilities. *Journal of Political Economy* **81**(3), 637–654.

Bodie, Z. (1983). Commodity futures as a hedge against inflation. *The Journal of Portfolio Management* **9**(3), 12–17.

Borsa Italiana (2009). *A new way of investing in commodities: ETC – exchange traded commodities*. Borsa Italiana Publications.

Bouchard, B. and E. Temam (2005). On the hedging of American options in discrete time markets with proportional transaction costs. *Electronic Journal of Probability* **10**(2000), 746–760.

Brennan, M. and E. Schwartz (1977). The valuation of American put options. *Journal of Finance* **32**(2), 449–462.

Brooks, N. (2008). Exchange traded commodities: commodity investing goes mainstream. *Alchemist – The London Bullion Market Association* **51**, 7–9.

Buetow, G. W. and B. J. Henderson (2012). An empirical analysis of exchange-traded funds. *Journal of Portfolio Management* **38**(4), 112–127.

Carr, P., H. Geman, and D. B. Madan (2001). Pricing and hedging in incomplete markets. *Journal of Financial Economics* **62**(1), 131–167.

Chalasani, P. and S. Jha (2001). Randomized stopping times and American option pricing with transaction costs. *Mathematical Finance* **11**(1), 33–77.

Charupat, N. and P. Miu (2011). The pricing and performance of leveraged exchange-traded funds. *Journal of Banking & Finance* **35**(4), 966–977.

Chen, A. (1970). A model of warrant pricing in a dynamic market. *The Journal of Finance* **25**(5), 1041–1059.

Chen, A. H. and J. W. Kensinger (1990). An analysis of market-index certificates of deposit. *Journal of Financial Services Research* **4**(2), 93–110.

Cheridito, P., F. Delbaen, and M. Kupper (2006). Dynamic monetary risk measures for bounded discrete-time processes. *Electronic Journal of Probability* **11**(3), 57–106.

Cheridito, P. and M. Kupper (2009). Recursiveness of indifference prices and translation-invariant preferences. *Mathematics and Financial Economics* **2**(3), 173–188.

— (2011). Composition of time-consistent dynamic monetary risk measures in discrete time. *International Journal of Theoretical and Applied Finance* **14**(01), 137–162.

Cheridito, P. and M. Stadje (2009). Time-inconsistency of VaR and time-consistent alternatives. *Finance Research Letters* **6**(1), 40–46.

Chu, P. K.-K. (2011). Study on the tracking errors and their determinants: Evidence from Hong Kong exchange traded funds. *Applied Financial Economics* **21**(5), 309–315.

Coleman, T. F., D. Levchenkov, and Y. Li (2007). Discrete hedging of American-type options using local risk minimization. *Journal of Banking & Finance* **31**(11), 3398–3419.

Collin-Dufresne, P. and J. Hugonnier (2007). Pricing and hedging in the presence of extraneous risks. *Stochastic Processes and their Applications* **117**(6), 742–765.

— (2013). Event risk, contingent claims and the temporal resolution of uncertainty. *Math Finan Econ* **8**(1), 29–69.

Constantinides, G. M. and S. Perrakis (2007). Stochastic dominance bounds on American option prices in markets with frictions. *Review of Finance* **11**(1), 71–115.

Constantinides, G. M. and T. Zariphopoulou (1999). Bounds on prices of contingent claims in an intertemporal economy with proportional transaction costs and general preferences. *Finance and Stochastics* **3**(3), 345–369.

— (2001). Bounds on derivative prices in an intertemporal setting with proportional transaction costs and multiple securities. *Mathematical Finance* **11**(3), 331–346.

Cox, J. C., S. A. Ross, and M. Rubinstein (1979). Option pricing: a simplified approach. *Journal of Financial Economics* **7**(3), 229–263.

Cvitanić, J., H. Pham, and N. Touzi (1999). Super-replication in stochastic volatility models under portfolio constraints. *Journal of Applied Probability* **36**(2), 523–545.

Davis, M. H. A. and J. M. C. Clark (1994). A note on super-replicating strategies. *Philosophical Transactions: Physical Sciences and Engineering* **347**(1684), 485–494.

Davis, M. H. A. (2006). Optimal Hedging with Basis Risk. *From Stochastic Calculus to Mathematical Finance*. Springer, 169–187.

Davis, M. H. A., V. G. Panas, and T. Zariphopoulou (1993). European option pricing with transaction costs. *SIAM Journal on Control and Optimization* **31**(2), 470–493.

Davis, M. H. A. and D. Yoshikawa (2015). A note on utility-based pricing in models with transaction costs. *Mathematics and Financial Economics* **9**(3), 231–245.

Davis, M. H. A. and T. Zariphopoulou (1995). American Options and Transaction Fees. *Mathematical Finance (The IMA Volumes in Mathematics and its Applications, Vol. 65)*. Springer, 47–61.

De Vallière, D., E. Denis, and Y. Kabanov (2008). Hedging of American options under transaction costs. *Finance and Stochastics* **13**(1), 105–119.

Delbaen, F., P. Grandits, T. Rheinländer, D. Samperi, M. Schweizer, and C. Stricker (2002). Exponential hedging and entropic penalties. *Mathematical Finance* **12**(2), 99–123.

Delbaen, F. and W. Schachermayer (1994). A general version of the fundamental theorem of asset pricing. *Mathematische Annalen* **300**(1), 463–520.

— (1998). The fundamental theorem of asset pricing for unbounded stochastic processes. *Mathematische Annalen* **312**(2), 215–250.

Delcoure, N. and M. Zhong (2007). On the premiums of iShares. *Journal of Empirical Finance* **14**(2), 168–195.

Detlefsen, K. and G. Scandolo (2005). Conditional and dynamic convex risk measures. *Finance and Stochastics* **9**(4), 539–561.

Deville, L. (2008). Exchange Traded Funds: History, Trading, and Research. *Handbook of Financial Engineering*. Ed. by C. Zopounidis, M. Doumpos, and P. Pardalos. Springer, 67–98.

D'Hondt, C. and J.-R. Giraud (2008). *Transaction cost analysis A-Z: A step towards best execution in the post-MiFID landscape*. EDHEC Publications.

Diavatopoulos, D., J. Felton, and C. Wright (2011). The indicative value-price puzzle in ETNs: liquidity constraints, information signaling, or an ineffective system for share creation? *The Journal of Investing* **20**(3), 25–39.

Doob, J. (1994). *Measure Theory*. Vol. 143. Graduate Texts in Mathematics Series. Springer.

Duffie, D., W. H. Fleming, H. M. Soner, and T. Zariphopoulou (1997). Hedging in incomplete markets with HARA utility. *Journal of Economic Dynamics and Control* **21**(4-5), 753–782.

Elton, E. J., M. J. Gruber, G. Comer, and K. Li (2002). Spiders: where are the bugs? *The Journal of Business* **75**(3), 453–472.

Engelke, L. and J. C. Yuen (2008). Types of Commodity Investments. *The Handbook of Commodity Investing*. Ed. by F. J. Fabozzi, R. Füss, and D. G. Kaiser. Hoboken (u.a.): John Wiley & Sons, Inc., 549–569.

Engle, R. F. and D. Sarkar (2006). Premiums-discounts and exchange traded funds. *The Journal of Derivatives* **13**(4), 27–45.

ETFS Commodity Securities Limited (2016). *Prospectus for the issue of ETFS Classic Commodity Securities*. Jersey.

ETFS Metal Securities Ltd. (2016). *Prospectus for the issue of ETFS Metal Securities*. Jersey.

Fabozzi, F. J., R. Füss, and D. G. Kaiser (2008). A Primer on Commodity Investing. *The Handbook of Commodity Investing*. Ed. by F. J. Fabozzi, R. Füss, and D. G. Kaiser. Hoboken (i.a.): John Wiley & Sons, Inc., 3–37.

Fassas, A. P. (2014). Tracking ability of ETFs: physical versus synthetic replication. *Index Investing* **5**(2), 9–20.

Föllmer, H. and A. Schied (2002). Convex measures of risk and trading constraints. *Finance and Stochastics* **6**(4), 429–447.

Frei, C. and M. Schweizer (2008). Exponential utility indifference valuation in two Brownian settings with stochastic correlation. *Advances in Applied Probability* **40**(2), 401–423.

— (2010). Exponential Utility Indifference Valuation in a General Semimartingale Model. *Optimality and Risk - Modern Trends in Mathematical Finance*. Springer, 49–86.

Frino, A. and D. R. Gallagher (2002). Is index performance achievable? An analysis of Australian equity index funds. *Abacus* **38**(2), 200–214.

Frittelli, M. (2000a). Introduction to a theory of value coherent with the no-arbitrage principle. *Finance and Stochastics* **4**(3), 275–297.

— (2000b). The minimal entropy martingale measure and the valuation problem in incomplete markets. *Mathematical Finance* **10**(1), 39–52.

Gallagher, D. R. and R. Segara (2006). The performance and trading characteristics of exchange-traded funds. *Journal of Investment Strategy* **1**(2), 49–60.

Gastineau, G. L. (2001). Exchange traded funds: an introduction. *The Journal of Portfolio Management* **27**(3), 88–96.

— (2010). *The Exchange-Traded Funds Manual*. 2. Hoboken (u.a.): John Wiley & Sons, Inc.

Gerer, J. and G. Dorfleitner (2016a). Optimal discrete hedging of American options using an integrated approach to options with complex embedded decisions. *SSRN eLibrary, included in this thesis as chapter 3*, 27–49.

— (2016b). Time consistent pricing of options with embedded decisions. *SSRN eLibrary, included in this thesis as chapter 2*, 5–27.

Geske, R. and H. E. Johnson (1984). The American put option valued analytically. *Journal of Finance* **39**(5), 1511–1524.

Gharakhani, M., F. Z. Fazlelahi, and S. J. Sadjadi (2014). A robust optimization approach for index tracking problem. *Journal of Computer Science* **10**(12), 2450–2463.

Giles, M. B. and R. Carter (2006). Convergence analysis of Crank–Nicolson and Rannacher time-marching. *The Journal of Computational Finance* **9**(4), 89–112.

Gobet, E. and N. Landon (2014). Almost sure optimal hedging strategy. *Ann. Appl. Probab.* **24**(4), 1652–1690.

Gorton, G. and K. G. Rouwenhorst (2006). Facts and fantasies about commodity futures. *Financial Analysts Journal* **62**(2), 47–68.

Grasselli, M. R. and T. R. Hurd (2007). Indifference pricing and hedging for volatility derivatives. *Applied Mathematical Finance* **14**(4), 303–317.

Grasselli, M. R. and V. Henderson (2009). Risk aversion and block exercise of executive stock options. *Journal of Economic Dynamics and Control* **33**(1), 109–127.

Gruber, M. J. (1996). Another puzzle: the growth in actively managed mutual funds. *The Journal of Finance* **51**(3), 783–810.

Grünbichler, A. and H. Wohlwend (2005). The valuation of structured products: empirical findings for the Swiss market. *Financial Markets and Portfolio Management* **19**(4), 361–380.

Guo, K. and T. Leung (2015). Understanding the Tracking Errors of Commodity Leveraged ETFs. *Commodities, Energy and Environmental Finance*. Ed. by R. Aïd, M. Ludkovski, and R. Sircar. New York, NY: Springer New York, 39–63.

Henderson, V. (2002). Valuation of claims on nontraded assets using utility maximization. *Mathematical Finance* **12**(4), 351–373.

— (2005). The impact of the market portfolio on the valuation, incentives and optimality of executive stock options. *Quantitative Finance* **5**(1), 35–47.

— (2007). Valuing the option to invest in an incomplete market. *Mathematics and Financial Economics* **1**(2), 103–128.

Henderson, V. and D. G. Hobson (2002a). Real options with constant relative risk aversion. *Journal of Economic Dynamics and Control* **27**(2), 329–355.

— (2002b). Substitute hedging. *RISK* **15**(5), 71–75.

— (2009). Utility indifference pricing: An overview. *Indifference pricing: theory and applications*. Ed. by R. Carmona. Princton University Press. Chap. 2, 44–73.

— (2011). Optimal liquidation of derivative portfolios. *Mathematical Finance* **21**(3), 365–382.

Henderson, V., J. Sun, and A. E. Whalley (2014). Portfolios of American options under general preferences: results and counterexamples. *Mathematical Finance* **24**(3), 533–566.

Hodges, S. D. and A. Neuberger (1989). Optimal replication of contingent claims under transaction costs. *The Review of Futures Markets* **8**(2), 222–239.

Hu, Y., P. Imkeller, and M. Müller (2005). Utility maximization in incomplete markets. *Annals of Applied Probability* **15**(3), 1691–1712.

Hugonnier, J., D. Kramkov, and W. Schachermayer (2005). On utility-based pricing of contingent claims in incomplete markets. *Mathematical Finance* **15**(2), 203–212.

Hyer, T., A. Lipton-Lifschitz, and D. Pugachevsky (1997). Passport to success. *Risk Magazine* **10**(9), 127–131.

Ibragimov, M., R. Ibragimov, and J. Walden (2015). *Heavy-tailed distributions and robustness in economics and finance*. Vol. 214. Lecture Notes in Statistics. Springer.

İlhan, A., M. Jonsson, and R. Sircar (2005). Optimal investment with derivative securities. *Finance and Stochastics* **9**(4), 585–595.

İlhan, A. and R. Sircar (2006). Optimal static–dynamic hedges for barrier options. *Mathematical Finance* **16**(2), 359–385.

Ito, K. and J. Toivanen (2009). Lagrange multiplier approach with optimized finite difference stencils for pricing American options under stochastic volatility. *SIAM J. Sci. Comput.* **31**(4), 2646–2664.

Jares, T. B. and A. M. Lavin (2004). Japan and Hong Kong exchange-traded funds (ETFs): discounts, returns, and trading strategies. *Journal of Financial Services Research* **25**(1), 57–69.

Jensen, M. C. (1967). The performance of mutual funds in the period 1945–1964. *The Journal of Finance* **23**(2), 389–416.

Johnson, W. F. (2009). Tracking errors of exchange traded funds. *J Asset Manag* **10**(4), 253–262.

Kabanov, Y. M. and C. Stricker (2002). On the optimal portfolio for the exponential utility maximization: remarks to the six-author paper. *Mathematical Finance* **12**(2), 125–134.

Kallsen, J. and C. Kühn (2004). Pricing derivatives of American and game type in incomplete markets. *Finance and Stochastics* **8**(2), 261–284.

— (2006). On utility-based derivative pricing with and without intermediate trades. *Statistics & Decisions* **24**(4/2006), 415–434.

Kallsen, J. and T. Rheinländer (2011). Asymptotic utility-based pricing and hedging for exponential utility. *Statistics & Decisions* **28**(1), 17–36.

Karatzas, I. (1988). On the pricing of American options. *Applied Mathematics and Optimization* **17**(1), 37–60.

— (1989). Optimization problems in the theory of continuous trading. *SIAM Journal on Control and Optimization* **27**(6), 1221–1259.

Karatzas, I. and S. G. Kou (1996). On the pricing of contingent claims under constraints. *Ann. Appl. Probab.* **6**(2), 321–369.

Kayali, M. M. and N. Ozkan (2012). Does the market misprice real sector ETFs in Turkey? *International Research Journal of Finance and Economics* **91**, 156–160.

Kostovetsky, L. (2003). Index mutual funds and exchange-traded funds. *The Journal of Portfolio Management* **29**(4), 80–92.

Kramkov, D. and J. Hugonnier (2004). Optimal investment with random endowments in incomplete markets. *Ann. Appl. Probab.* **14**(2), 845–864.

Kramkov, D. and M. Sîrbu (2006). Sensitivity analysis of utility-based prices and risk-tolerance wealth processes. *Annals of Applied Probability* **16**(4), 2140–2194.

Lan, S., S. Mercado, and S. Levitt (2013). 2012 ETF review & 2013 outlook: Record inflows drive global ETP assets to near $2 trillion. *Deutsche Bank Markets Research*.

Lang, S. E. (2009). *Exchange Traded Funds – Erfolgsgeschichte und Zukunftsaussichten*. Duisburg (i.a.): WiKu.

Lehmann, B. N. and D. M. Modest (1987). Mutual fund performance evaluation: a comparison of benchmarks and benchmark comparisons. *The Journal of Finance* **42**(2), 233–265.

Leung, T. and R. Sircar (2009a). Accounting for risk aversion, vesting, job termination risk and multiple exercises in valuation of employee stock options. *Mathematical Finance* **19**(1), 99–128.

Leung, T. and R. Sircar (2009b). Exponential hedging with optimal stopping and application to employee stock option valuation. *SIAM Journal on Control and Optimization* **48**(3), 1422–1451.

Leung, T. and B. Ward (2015). The golden target: Analyzing the tracking performance of leveraged gold ETFs. *Studies in Economics & Finance* **32**(3), 278–297.

Lin, A. and A. Chou (2006). The tracking error and premium/discount of Taiwan's first exchange traded fund. *Web Journal of Chinese Management Review* **9**(3), 1–21.

Malamud, S., E. Trubowitz, and M. V. Wüthrich (2013). Indifference pricing for CRRA utilities. *Math Finan Econ* **7**(3), 247–280.

Malkiel, B. G. (1995). Returns from investing in equity mutual funds 1971 to 1991. *The Journal of Finance* **50**(2), 549–572.

Mania, M. and M. Schweizer (2005). Dynamic exponential utility indifference valuation. *Annals of Applied Probability* **15**(3), 2113–2143.

Mankiewicz, C. (2009). Aktives vs. passives Management von Commodity-Investments —Sind passive Indexinvestments der geeignete Ansatz für Pensionskassen? *e-Journal of Practical Business Research, Sonderausgabe Performance (01/2009)*, 1–26.

McKean, H. P. (1965). Appendix: a free-boundary problem for the heat-equation arising from a problem of mathematical economics. *Industrial Management Review* **6**(2), 32–39.

Merton, R. C. (1969). Lifetime portfolio selection under uncertainty: the continuous time case. *Review of Economics and Statistics* **51**(3), 247–257.

— (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory* **3**(4), 373–413.

— (1973a). Erratum. *Journal of Economic Theory* **6**(2), 213–214.

— (1973b). Theory of rational option pricing. *Bell Journal of Economics and Management Science* **4**(1), 141–183.

Mohamed, B. (1994). Simulations of transaction costs and optimal rehedging. *Applied Mathematical Finance* **1**(1), 49–62.

Monoyios, M. (2003). Efficient option pricing with transaction costs. *Journal of Computational Finance* **7**(1), 107–128.

— (2004a). Option pricing with transaction costs using a markov chain approximation. *Journal of Economic Dynamics and Control* **28**(5). Financial decision models in a dynamical setting, 889–913.

— (2004b). Performance of utility-based strategies for hedging basis risk. *Quantitative Finance* **4**(3), 245–255.

— (2006). Characterisation of optimal dual measures via distortion. *Decisions in Economics and Finance* **29**(2), 95–119.

Muck, M. (2006). Where should you buy your options? The pricing of exchange-traded certificates and OTC derivatives in Germany. *The Journal of Derivatives* **14**(1), 82–96.

Musiela, M. and T. Zariphopoulou (2004a). A valuation algorithm for indifference prices in incomplete markets. *Finance and Stochastics* **8**(3), 399–414.

— (2004b). An example of indifference prices under exponential preferences. *Finance and Stochastics* **8**(2), 229–239.

Myneni, R. (1992). The pricing of the American option. *Annals of Applied Probability* **2**(1), 1–23.

Owen, M. P. and G. Žitković (2009). Optimal investment with an unbounded random endowment and utility-based pricing. *Mathematical Finance* **19**(1), 129–159.

Parkinson, M. (1977). Option pricing: the American put. *Journal of Business* **50**(1), 21–36.

Pham, H. (2009). *Continuous-time stochastic control and optimization with financial applications*. Ed. by B. Rozovskii and G. Grimmett. Vol. 61. Stochastic Modelling and Applied Probability. Springer Berlin Heidelberg.

Plante, J.-F. and M. Roberge (2007). The passive approach to commodity investing. *Journal of Financial Planning* **37**(4), 66–75.

Ramaswamy, S. (2011). Market structures and systemic risks of exchange-traded funds. *BIS Working Paper No. 343*, 1–17.

Rannacher, R. (1984). Finite element solution of diffusion problems with irregular data. *Numerische Mathematik* **43**(2), 309–327.

Rheinländer, T. and G. Steiger (2010). Utility indifference hedging with exponential additive processes. *Asia-Pacific Financial Markets* **17**(2), 151–169.

Rockafellar, R. T. (1970). *Convex Analysis*. Princeton, New Jersey: Princteon University Press, 468.

Rogers, L. C. G. and J. Scheinkman (2007). Optimal exercise of executive stock options. *Finance and Stochastics* **11**(3), 357–372.

Roll, R. (1992). A mean/variance analysis of tracking error. *The Journal of Portfolio Management* **18**(4), 13–22.

Rompotis, G. G. (2008). Performance and trading characteristics of German passively managed ETFs. *International Research Journal of Finance and Economics* **15**, 218–231.

— (2011a). Active vs. passive management: new evidence from exchange traded funds. *International Review of Applied Financial Issues and Economics* **3**(1), 169–186.

— (2011b). Predictable patterns in ETFs' return and tracking error. *Studies in Economics & Finance* **28**(1), 14–35.

Rouge, R. and N. El Karoui (2000). Pricing via utility maximization and entropy. *Mathematical Finance* **10**(2), 259–276.

Roux, A. and T. Zastawniak (2009). American options under proportional transaction costs: pricing, hedging and stopping algorithms for long and short positions. *Acta Applicandae Mathematicae* **106**(2), 199–228.

— (2014). American options with gradual exercise under proportional transaction costs. *International Journal of Theoretical and Applied Finance* **17**(08), 1450052.

— (2016). American and bermudan options in currency markets with proportional transaction costs. *Acta Applicandae Mathematicae* **141**(1), 187–225.

Rudolf, M., H.-J. Wolter, and H. Zimmermann (1999). A linear model for tracking error minimization. *Journal of Banking & Finance* **23**(1), 85–103.

Samuelson, P. A. (1965). Rational theory of warrant pricing. *Industrial Management Review* **6**(2), 13–31.

Schmidhammer, C., S. Lobe, and K. Röder (2010). Intraday pricing of ETFs and certificates replicating the German DAX index. *Review of Managerial Science* **5**(4), 337–351.

Sethi, S. P. and M. Taksar (1988). A note on Merton's "Optimum consumption and portfolio rules in a continuous-time model". *Journal of Economic Theory* **46**(2), 395–401.

Shin, S. and G. Soydemir (2010). Exchange-traded funds, persistence in tracking errors and information dissemination. *Journal of Multinational Financial Management* **20**(4/5), 214–234.

Soner, H. M., S. E. Shreve, and J. Cvitanic (1995). There is no nontrivial hedging portfolio for option pricing with transaction costs. *Annals of Applied Probability* **5**(2), 327–355.

Stoimenov, P. A. and S. Wilkens (2005). Are structured products 'fairly' priced? An analysis of the German market for equity-linked instruments. *Journal of Banking & Finance* **29**(12), 2971–2993.

Stoll, H. R. and R. E. Whaley (2010). Commodity index investing and commodity futures prices. *Journal of Applied Finance* **20**(1), 7–46.

Svensson, L. E. and I. M. Werner (1993). Nontraded assets in incomplete markets: pricing and portfolio choice. *European Economic Review* **37**(5). Special Issue On Finance, 1149–1168.

Tokarz, K. and T. Zastawniak (2006). American contingent claims under small proportional transaction costs. *Journal of Mathematical Economics* **43**(1), 65–85.

Tzvetkova, R. (2005). *The Implementation of Exchange Traded Funds in the European Market*. Bamberg: Difo-Druck.

Wallmeier, M. and M. Diethelm (2009). Market pricing of exotic structured products: the case of multi-asset barrier reverse convertibles in switzerland. *The Journal of Derivatives* **17**(2), 59–72.

Welch, B. L. (1951). On the comparison of several mean values: an alternative approach. *Biometrika* **38**(3/4), 330–336.

White, H. (1980). A heteroskedasticity-consistent covariance matrix estimator and a direct test for heteroskedasticity. *Econometrica* **48**(4), 817–838.

Wilkens, S., C. Erner, and K. Röder (2003). The pricing of structured products in Germany. *The Journal of Derivatives* **11**(1), 55–69.

Wright, C., D. Diavatopoulos, and J. Felton (2010). Exchange-traded notes: an introduction. *The Journal of Investing* **19**(2), 27–37.