Calculating the tensor charge of the nucleon using lattice QCD

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0 Notations

Throughout the work summation conventions are used for spatial and Minkowski indices and indices of the adjoint SU(3) representation.
Color and Dirac indices will be omitted if possible.

The following notation will be used:

Natural units	$c = \hbar = 1$	(1)
Contravariant vector	$x^\mu = \left(x^0, x^1, x^2, x^3\right) = (t, x, y, z) = (t, \vec{x})$	(2)
Metric tensor	$g^{\mu\nu} = g_{\mu\nu} = \text{diag}\,(1, -1, -1, -1)$	(3)
Covariant vector	$x_\mu = g_{\mu\nu} x^\nu = (t, -\vec{x})$	(4)
Scalar product	$x \cdot y = x^\mu y_\mu = x^0 y^0 - \vec{x} \cdot \vec{y}$	(5)
Four-momentum	$p^\mu = (E, -\vec{p}) = i\partial^\mu = i\frac{\partial}{\partial x_\mu} = i\left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)$	(6)
Four-spin	$(0, \vec{s}) \xrightarrow{\text{Lorentz boost}} s^\mu(\vec{p})$, with $p^\mu s_\mu(\vec{p}) = 0$	(7)
	and $s^2 = -m^2$	
Dirac matrices	$\gamma^\mu = (\gamma^0, \vec{\gamma})$, with $\{\gamma^\mu, \gamma^\nu\} = 2\, g^{\mu\nu}$	(8)
	$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$	(9)
Feynman dagger	$\not{a} = a_\mu \gamma^\mu$	(10)
Quark spinor	$\psi_f^{\alpha,c}(x)$, with $x \in \mathbb{R}^4$, color index $c = 1, 2, 3$, Dirac spinor index $\alpha = 1, 2, 3, 4$ and flavor f	(11)
SU(3) generators	T^a, with $a = 1, \ldots, 8$	(12)
Four potential	$\mathrm{A}_\mu(x) = A^a{}_\mu(x)\, \mathrm{T}^a$	(13)
Covariant derivative	$\mathrm{D}_\mu = \partial_\mu - ig\mathrm{A}_\mu$	(14)
Gauge transporter	$\mathrm{U}(x, y) = \mathcal{P}\left\{\exp\left[ig\int_x^y \mathrm{d}z_\mu\, \mathrm{A}^\mu(z)\right]\right\}$[1]	(15)

[1] $\mathcal{P}$ denotes path-ordering

Dimensionless gauge coupling parameter	g	(16)
Gauge transformation	$\mathcal{G}(x) = e^{ig\theta^a(x)\mathrm{T}^a}$, with $\theta^a(x) \in \mathbb{R}$	(17)
	$\psi(x) \to \mathcal{G}(x)\psi(x)$	(18)
	$\mathrm{A}_\mu(x) \to \mathcal{G}(x)(\mathrm{A}_\mu(x) - \frac{i}{g}\partial_\mu)\mathcal{G}(x)^{-1}$	(19)
	$\mathrm{D}_\mu(x) \to \mathcal{G}(x)\mathrm{D}(x)\mathcal{G}(x)^{-1}$	(20)
	$\mathrm{U}(x,y) \to \mathcal{G}(x)\mathrm{U}(x,y)\mathcal{G}(y)^{-1}$	(21)
Dirac operator	$K(x) = (i\not{\mathrm{D}}(x) - m)$	(22)
Field strength tensor	$\mathrm{F}_{\mu\nu} = \frac{i}{g}\left[\mathrm{D}_\mu, \mathrm{D}_\nu\right] = \partial_\mu \mathrm{A}_\nu - \partial_\nu \mathrm{A}_\mu - ig\left[\mathrm{A}_\mu, \mathrm{A}_\nu\right]$	(23)
Euclidean vector	$x^{\mathrm{E}}_\mu = (x^{\mathrm{E}}_1, x^{\mathrm{E}}_2, x^{\mathrm{E}}_3, x^{\mathrm{E}}_4) = (\vec{x}, it)$	(24)
Euclidean covariant derivative	$\mathrm{D}^{\mathrm{E}}_\mu = \partial^{\mathrm{E}}_\mu - ig\mathrm{A}^{\mathrm{E}}_\mu = (\mathbf{D}, -i\mathrm{D}^0)$	(25)
Euclidean Dirac matrices	$\gamma^{\mathrm{E}}_\mu = (-i\boldsymbol{\gamma}, \gamma^0)$, with $\left\{\gamma^{\mathrm{E}}_\mu, \gamma^{\mathrm{E}}_\nu\right\} = 2\,\delta^{\mu\nu}$	(26)
Euclidean Dirac operator	$\mathrm{K}(x)^{\mathrm{E}} = \gamma^{\mathrm{E}}_\mu \mathrm{D}^{\mathrm{E}}_\mu + m$	(27)
Parity projectors	$P_\pm = \dfrac{1 \pm \gamma_0}{2} = \dfrac{1 \pm \gamma^{\mathrm{E}}_4}{2},\ P_+ + P_- = \mathbb{1},\ P^2_\pm = P_\pm$	(28)
Spin projectors	$P(\vec{s}) = \dfrac{1 + \gamma_5 \not{s}/m}{2} = \dfrac{1 + i\gamma^{\mathrm{E}}_5 \not{s}^{\mathrm{E}}/m}{2}$	(29)
	$P(\vec{s}) + P(-\vec{s}) = \mathbb{1}$ and $P(\vec{s})^2 = P(\vec{s})$	(30)
Positive energy spinors	$u(\vec{s},\vec{p})\bar{u}(\vec{s},\vec{p}) = \left(-i\not{p}^{\mathrm{E}} + m\right) P(\vec{s})$	(31)
Negative energy spinors	$v(\vec{s},\vec{p})\bar{v}(\vec{s},\vec{p}) = \left(-i\not{p}^{\mathrm{E}} - m\right) P(\vec{s})$	(32)
Lattice constant	a	(33)
Transition to dimensionless lattice quantities	$m_{\mathrm{L}} = am,\ \ \psi_{\mathrm{L}}(n_\mu) = a^{3/2}\psi(x_\mu)$	(34)

1 Lattice QCD

The usual way to achieve analytic results from quantum field theories included in the standard model, is to perform a perturbative expansion. In contrast to QED, where predictions fit the experiments at all physically accessible energy scales, the large coupling constant of QCD (≈ 1) at low energy scales practically forbids this approach. Unfortunately the description of baryonic matter, that our world is made of, also lies far away form the perturbative domain. In order to theoretically investigate the bound states formed via QCD interactions, that make up the most common particles of the universe, a non-perturbative approach was needed. Without employing a model but instead directly deriving it from the fundamental theory of QCD, the solution to this dilemma was lattice QCD.

Lattice QCD is basically the numerical calculation of the path integrals arising from continuum QCD.

1.1 Path integral formalism

In the path integral formalism the vacuum expectation value of an operator $\mathcal{O}\big[\bar{\psi}, \psi, A\big]$ can be calculated through the path integral

$$\big\langle \mathcal{O}\big[\bar{\psi}, \psi, A\big] \big\rangle = \frac{\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A \; \mathcal{O}\big[\bar{\psi}, \psi, A\big] \;\; e^{i\mathcal{S}[\bar{\psi}, \psi, A]}}{\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A \; e^{i\mathcal{S}[\bar{\psi}, \psi, A]}} \tag{35}$$

$\mathcal{S}$ denotes the action defined by the integral over the Lagrangian density of QCD

$$\mathcal{S} = \int \mathrm{d}^4x \; \mathcal{L}(x) = \int \mathrm{d}^4x \left\{ \sum_f \bar{\psi}_f (i \not{D} - m_f) \psi_f - \frac{1}{2} \mathrm{tr}[\mathrm{F}_{\mu\nu} \mathrm{F}^{\mu\nu}] \right\} \tag{36}$$

The integration measures (e.g. $\mathcal{D}\psi$) are defined as the multi-dimensional integral over all degrees of freedom of the fields (cp. eqs. (11) and (13)).

Before further evaluation of this integral one switches to Euclidean time. This can be implemented by using an imaginary time component in the spacetime integrals and parameters.

$$t \to t = -i\tau, \text{ with } \tau \in \mathbb{R} \tag{37}$$

This way the action $\mathcal{S}$ becomes purely imaginary and the exponential weight factor becomes real-valued and negative

$$e^{i\mathcal{S}} = e^{-\mathcal{S}_\mathrm{E}}, \tag{38}$$

and can be more conveniently expressed using a real-valued Euclidean action of the following form.

$$\mathcal{S}_\mathrm{E} = \int \mathrm{d}^4x^\mathrm{E} \left\{ \sum_f \bar{\psi}_f (\gamma_\mu^\mathrm{E} \mathrm{D}_\mu^\mathrm{E} + m_f) \psi_f + \frac{1}{2} \mathrm{tr}\big[\mathrm{F}_{\mu\nu}^\mathrm{E} \mathrm{F}_{\mu\nu}^\mathrm{E}\big] \right\} \tag{39}$$

This step was necessary to get rid of the rapidly oscillating factor e^{iS}, as numerical calculations only become feasible with a real-valued damping factor e^{-S_E} that is bounded from above.
From now on we will work in Euclidean spacetime and all labels E will be dropped, unless stated differently.

The next step towards a numerical calculation of the infinite-dimensional integral is to approximate it by a finite dimensional one by limiting the degrees of freedom of the fields. This can be done by replacing the continuous spacetime with a spacetime lattice and restricting the quark fields to only exist on lattice points

$$x = a(n_1, n_2, n_3, n_4), \text{ with } n_\mu \in \{0, 1, \ldots, N_\mu - 1\}\,. \tag{40}$$

Now the form of the path integral is suitable for the use of Monte-Carlo methods.
For a more detailed introduction and references see [1] and [2].

1.2 Fermions on the lattice

In view of the anti-commuting character of the fermion field operator, it is not possible to use regular complex numbers in the path integral, as they are commuting. Instead, the 'classical' fermion fields need to be Grassmann degrees of freedom, hence anti-commuting numbers. This also requires anti-periodic boundary conditions in time. Spatial boundary condition are chosen to be periodic.
From now on we use dimensionless lattice variables and drop the labels 'L' from eq. 34.
To describe fermions on the lattice, a lattice version of the Dirac operator is needed, that closely resembles the continuum operator.
In continuum QCD the action of the Dirac operator on a field can be described by an integral kernel whose lattice equivalent is the lattice Dirac operator that needs to be found:

$$\mathrm{K}(x)\psi(x) = \int \mathrm{d}^4 y\, \mathrm{K}(x, y)\psi(y) \to \sum_m \mathrm{K}(n, m)\psi(m) \tag{41}$$

It turns out that it is hard to find an operator that possesses all desired properties. There is even mathematical proof that such an operator does not exist (Nielsen-Ninomiya No-Go theorem). So over time, many different operators have been developed that satisfies only some of the properties of the continuum operator.
One popular approach are the 'Chirally Improved fermions' (CI-fermions). The chirally improved Dirac operator K is an approximate solution to the Ginsparg-Wilson relation

$$\left\{\mathrm{K}^{-1}, \gamma_5\right\} = a\,\gamma_5. \tag{42}$$

Hence it violates the chiral symmetry only in a minimal and controled way, until its full recovery in the continuum limit.
To render the fermionic action gauge invariant, $\mathrm{K}(n, m)$ needs to transform like a gauge transporter (eq. 21). So the most general ansatz for a lattice Dirac operator is a

linear combination of Clifford algebra generators Γ^α and gauge transporters $\mathrm{U}_p(x,y)$ that connect x and y via the path p:

$$\mathrm{K}(y,x) = \sum_{\alpha}^{16} \sum_{p \in \mathcal{P}_{x,y}} c_\alpha^p \, \Gamma^\alpha \, \mathrm{U}_p(x,y) \tag{43}$$

Now all the requirements that the Dirac operator should fulfill (Ginsparg-Wilson relation; invariance under translation, rotation, charge conjugation and parity; γ_5-hermiticity; correct continuum limit) can be translated into equations for the coefficients. These equations are solved only for a certain number of coefficients, as every additional coefficient drives up the numerical cost of a numerical calculation. On the other hand, the more coefficients are included, the better becomes the approximation to the Ginsparg-Wilson relation. Due to the locality enforced by this condition, the importance of coefficients decreases with the length of the corresponding path.
All the results of this work were obtained from calculations using CI fermions. The coefficients that were used can be found in Appendix D of [1].

1.3 Gluons on the lattice

By the introduction of a discrete lattice the degrees of freedom of the gauge fields ${A^a}_\mu$ also are reduced to a finite number. In contrast to the fermion fields, that where restricted to live on the lattice sites, the gluon fields need to be handled differently.
As can be seen in eq. (43), the gauge field only occurs in the action in terms of the gauge transporters $U(x,y)$, thus only the values of line integrals over ${A^a}_\mu$ between to lattice points are of importance. Accounting for this fact one says that the gauge field lives on the links of neighboring lattice sites.
The values ${U^a}_\mu(n)$ with

$${U^a}_\mu(n) \, \mathrm{T}^a \equiv \mathrm{U}(n, n+\hat{\mu}) = \mathrm{e}^{iga\bar{\mathrm{A}}_\mu} \tag{44}$$

are called the 'link variable'. Here $\hat{\mu}$ is the unit vector in μ direction and $\bar{\mathrm{A}}_\mu$ denotes the average value on the link between lattice point n and $n+\hat{\mu}$.
In analogy to continuum QCD one endows the gauge fields with physical reality by adding a gauge invariant kinetic term to the Lagrangian.
The quantity tr $\mathrm{U}_p(n,n)$ for any closed path p is gauge invariant. Lüscher and Weisz proposed the combination of squares S, rectangles R and parallelograms P as paths:

$$\begin{aligned} S_{\text{Lüscher, Weisz}} = &\sum_S \beta_S \left(1 - \frac{1}{N} \mathrm{Re}\left\{\text{tr } \mathrm{U}_S\right\}\right) + \\ &\sum_R \beta_R \left(1 - \frac{1}{N} \mathrm{Re}\left\{\text{tr } \mathrm{U}_R\right\}\right) + \\ &\sum_P \beta_P \left(1 - \frac{1}{N} \mathrm{Re}\left\{\text{tr } \mathrm{U}_P\right\}\right), \end{aligned} \tag{45}$$

where $N = 3$ (for SU(3)) is the total number of sites and $\beta_S = 2N/g^2$.
Historically, Lüscher and Weisz proposed their gauge action to get rid of $\mathcal{O}(a^2)$ discretization errors occurring in Wilson's gauge action that only uses squares. So the

values of β_R and β_P need to be set to a specific value in order to cancel those errors. They can be stated in terms of the expectation value of the plaquette variable u_0^4:

$$u_0^4 = \frac{1}{3}\text{Re tr}\left\langle \mathrm{U_{plaq}} \right\rangle, \tag{46}$$

$$\alpha = -\frac{\ln u_0^4}{3.06839}, \tag{47}$$

$$\beta_R = -\frac{\beta_S}{20u_0^2}(1 + 0.4805\alpha), \tag{48}$$

$$\beta_P = -\frac{\beta_S}{u_0^2}0.03325\alpha \tag{49}$$

All the results of this work were obtained from calculations using the Lüscher and Weisz gauge action with with these coefficients.
For references see [1].

1.4 Evaluation of the path integral

The fermionic Lagrange density is a bilinear form in the fields. By following the rules for integration over Grassmann variables, the following integrals can be solved analytically:

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\; e^{-\bar{\psi}\mathrm{K}[A]\psi} = \det \mathrm{K}[A] \tag{50}$$

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\; \psi^{\alpha_1}\cdots\psi^{\alpha_j}\bar{\psi}^{\beta_1}\cdots\bar{\psi}^{\beta_j}\; e^{-\bar{\psi}\mathrm{K}[A]\psi} = \xi_j\,(\det \mathrm{K}[A])\sum_P (-1)^{\sigma_P}\mathrm{K}[A]^{-1}_{\alpha_1\beta_{P_1}}\cdots\mathrm{K}[A]^{-1}_{\alpha_j\beta_{P_j}} \tag{51}$$

where $\xi_j = (-1)^{j(j-1)/2}$, α symbolically stands for all parameters of the fermion field (Dirac, color, spacetime coordinate), the sum extends over all permutations and $(-1)^{\sigma_P}$ is the signum of the permutation:

$$P : \begin{pmatrix} \beta_1 & \beta_2 & \cdots & \beta_j \\ \beta_{P_1} & \beta_{P_2} & \cdots & \beta_{P_j} \end{pmatrix} \tag{52}$$

Now consider a non-interacting theory of fermions on a fixed gauge field background. In this case the path integral representation of n-point functions contains no integration over gauge fields and only the fermionic action is used:

$$\begin{aligned} \left\langle \psi^{\alpha_1}\cdots\psi^{\alpha_j}\bar{\psi}^{\beta_1}\cdots\bar{\psi}^{\beta_j}\right\rangle_A &= \frac{\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\; \psi^{\alpha_1}\cdots\psi^{\alpha_j}\bar{\psi}^{\beta_1}\cdots\bar{\psi}^{\beta_j} e^{-\mathcal{S}_F[\bar{\psi},\psi,A]}}{\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\; e^{-\mathcal{S}_F[\bar{\psi},\psi,A]}} \\ &\overset{(50)(51)}{=} \xi_j \sum_P (-1)^{\sigma_P}\mathrm{K}[A]^{-1}_{\alpha_1\beta_{P_1}}\cdots\mathrm{K}[A]^{-1}_{\alpha_j\beta_{P_j}} \end{aligned} \tag{53}$$

Eq. (53) is nothing else than the path integral derivation of Wick's theorem and the fact that n-point functions of non-interacting fermions can be expressed in terms of

the inverse of the Dirac operator.
This result can be used to evaluate further the integrals of the full interacting theory. First it should be mentioned that in the following it is sufficient to treat operators of the form

$$\mathcal{O}\big[\bar{\psi},\psi,A\big] = \psi^{\alpha_1}\cdots\psi^{\alpha_j}\bar{\psi}^{\beta_1}\cdots\bar{\psi}^{\beta_j}\;\mathcal{O}_G[A]\,, \tag{54}$$

as they can be used to construct any operator, i.e. a general functional in the fermion fields. Starting from eq. (35), the path integral expression of its vacuum expectation value can be restated in the following way

$$\big\langle\psi^{\alpha_1}\cdots\psi^{\alpha_j}\bar{\psi}^{\beta_1}\cdots\bar{\psi}^{\beta_j}\;\mathcal{O}_G[A]\big\rangle \tag{55}$$

$$= \frac{\int\mathcal{D}A\;\mathcal{O}_G[A]\;e^{-\mathcal{S}_G[A]}\left(\int\mathcal{D}\psi\mathcal{D}\bar{\psi}\;\psi^{\alpha_1}\cdots\psi^{\alpha_j}\bar{\psi}^{\beta_1}\cdots\bar{\psi}^{\beta_j}\;e^{-\mathcal{S}_F[\bar{\psi},\psi,A]}\right)}{\int\mathcal{D}A\;e^{-\mathcal{S}_G[A]}\left(\int\mathcal{D}\psi\mathcal{D}\bar{\psi}\;e^{i\mathcal{S}_F[\bar{\psi},\psi,A]}\right)} \tag{56}$$

$$= \frac{\int\mathcal{D}A\;\mathcal{O}_G[A]\;e^{-\mathcal{S}_G[A]}\left\langle\psi^{\alpha_1}\cdots\psi^{\alpha_j}\bar{\psi}^{\beta_1}\cdots\bar{\psi}^{\beta_j}\right\rangle_A\det\mathrm{K}[A]}{\int\mathcal{D}A\;e^{-\mathcal{S}_G[A]}\det\mathrm{K}[A]} \tag{57}$$

Lattice calculations that evaluate the determinants are called dynamic calculations. But as the numerical cost of the evaluation of the determinant of a $12N\times12N$ matrix is extremely high, some use the so called quenched approximation:

$$\det\mathrm{K}[A] = \text{const.} \tag{58}$$

This uncontrolled simplification of QCD has the effect of an infinite sea quark mass and neglecting the internal quark loops.
To actually numerically evaluate the integrals in eq. (57), Monte-Carlo methods are employed. Basically, instead of performing the sum over all possible field configurations, a sample of configurations is generated with a certain probability distribution and the integral simply becomes the average over the sample.
Within the quenched approximation eq. (57) with the use of eq. (53) becomes after statistical sampling with sample $\{A_i\}_{1\le i\le n}$ and probability distribution $e^{-S_G[A]}$:

$$\big\langle\psi^{\alpha_1}\cdots\psi^{\alpha_j}\bar{\psi}^{\beta_1}\cdots\bar{\psi}^{\beta_j}\;\mathcal{O}_G[A]\big\rangle = \frac{1}{n}\sum_i^n\;\mathcal{O}_G[A_i]\,\xi_j\sum_P(-1)^{\sigma_P}\mathrm{K}[A_i]^{-1}_{\alpha_1\beta_{P_1}}\cdots\mathrm{K}[A_i]^{-1}_{\alpha_j\beta_{P_j}} \tag{59}$$

All the results of this work were obtained in quenched approximation using formula (59).
For a more detailed introduction and references see [1] and [2].

1.5 Setting the scale

The theory formulated so far is scale invariant. The parameters, gauge coupling g or β, and the quark lattice masses, which are products of the lattice constant and the physical mass, are all dimensionless. Hence all results obtained from calculations are dimensionless. But we know that the physics of our universe posses a scale

that amounts to the typical size and mass of bound quark states, what is known as conformal anomaly.
The fact that the plain SU(3) Yang-Mills theory is already an interacting theory results in an interesting situation. Once the dimensionless gauge coupling parameter is specified, the regulator of the theory (e.g. the lattice constant a) is also implicitly specified.
'Setting the scale' denotes the process of determining this value. On the lattice this is in general done by selecting a quantity that has been measured in experiments and whose lattice equivalent only depends on the gauge coupling. Then the value of a can be determined by matching the dimensionless lattice result to agree with a dimensionless combination of a and the experimental value.
For example, the string tension σ of the static quark-quark potential can be used:

$$a^2\sigma_{\text{experiment}} \stackrel{!}{=} \sigma_{\text{lattice}} \rightarrow a \tag{60}$$

Sommer introduced a method that uses the characteristic length scale r_0 of the static quark potential to set the scale.
On typically uses a value $r_0 < 0.5\,\text{fm}$.
For a more detailed introduction and references see [1] and [2].

1.6 Chiral and continuum extrapolation

To obtain results that can be compared to continuum calculations or experiments a so called continuum extrapolation needs to be performed. For this purpose lattice results for different values of a (i.e. different values of β) are necessary.

Another problem is the huge amount of computer time that is needed to perform simulations at realistic quark masses. To circumvent this, on performs several simulations for higher masses and fits them to predictions of chiral perturbation theory for the physical masses, to obtain the physical results.

For a more detailed introduction and references see [1].

2 Moments of structure functions

Structure functions are used to parameterize unknown parts of the hadronic scattering tensor accounting for the unknown inner structure of hadrons.
Especially interesting are the proton and neutron structure functions. Within the parton model these structure functions can be expressed by the quark and gluon distribution functions, $q_\sigma(x)$ and $g_\sigma(x)$, resp. These functions can be interpreted as the probability to find a quark in the nucleon that carries the momentum fraction x

of the nucleon[2] and is polarized parallel/antiparallel with respect to the nucleon's spin. (This is denoted by $\sigma = \uparrow / \downarrow$ for longitudinal and by $\sigma = \perp / \top$ for transversal polarized nucleons)
Unfortunately, a direct theoretical computation of structure functions does not seem to be possible [3]. However, moments of distribution functions are related via the operator product expansion (OPE) to matrix elements of local operators, that can be calculated on the lattice.
The structure function arising in the Drell-Yan process with transverse polarized proton and anti-proton is called the transversity. The n-th moment of the transversity structure function is defined as

$$t_n^{(q)}(\mu) = \int_0^1 \mathrm{d}x\, x^{n-1} \left[\delta q(x,\mu) - (-1)^{n-1} \delta \bar{q}(x,\mu)\right], \text{ with } \delta q = q_\perp - q_\top \tag{61}$$

and can be related to the expectation value of the operator[3]

$$\mathcal{O}_{\sigma q}^{\mu_1 \cdots \mu_n} = \sum_{\vec{x}} i^{n-1} \bar{q}(\vec{x}) \sigma^{\mu_1 \mu_2} \gamma_5 \overleftrightarrow{\mathrm{D}}^{\mu_3} \cdots \overleftrightarrow{\mathrm{D}}^{\mu_n} q(\vec{x}) \tag{62}$$

by

$$\left\langle N_{\vec{s},\vec{p}} \middle| \mathcal{O}_{\sigma q}^{\mu_1 \{\mu_2 \cdots \mu_n\}} - \mathrm{tr} \middle| N_{\vec{s},\vec{p}} \right\rangle = 2 \frac{t_{n-1}^{(q)}}{m_N} \left[\left(s^{\mu_1} p^{\{\mu_2} - p^{\mu_1} s^{\{\mu_2} \right) p^{\mu_3} \cdots p^{\mu_n\}} - \mathrm{tr} \right]. \tag{63}$$

This and similar relations for other structure functions can be found in [4].
The nucleon states $|N_{\vec{s},\vec{p}}\rangle$ are continuum normalized:

$$\langle N_{\vec{s},\vec{p}} | N_{\vec{s},\vec{q}} \rangle = 2E_{\vec{p}} (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \tag{64}$$

The braces around Lorentz indices state that the expression has to be symmetrized with respect to the Lorentz indices. The subtraction of traces is to be performed in a way that the contraction of any two Lorentz indices vanishes:

$$g_{\alpha\beta}\, \mathcal{O}^{\mu_1 \cdots \alpha \cdots \beta \cdots \mu_n} = 0 \tag{65}$$

The lowest moment $t_1^{(q)}$ is called the tensor charge. The aim of this work is to calculate the non-singlet combination

$$t_1^{(u-d)} = t_1^{(u)} - t_1^{(d)} \tag{66}$$

on the lattice. This combination is chosen because the consequential cancellation of the so called quark line disconnected term leads to significant computation time savings. See section 2.2 for more detail.

[2] This is to be understood in the infinite momentum frame and with $\vec{p}_{\text{quark}} = x\, \vec{p}_{\text{nucleon}}$
[3] $\overleftrightarrow{\mathrm{D}}^{\mu} = \frac{1}{2} \left(\overrightarrow{\mathrm{D}}^{\mu} - \overleftarrow{\mathrm{D}}^{\mu} \right)$

2.1 Determination of matrix elements

The problem connected with the direct evaluation of the nucleon matrix element in (63) is our ignorance of the nucleon wave function. But what we know is that the nucleon is the lowest energy state made up of three valence quarks (uud or udd, resp.) with positive parity and forming a color singlet. It is possible to construct an operator that has the same quantum numbers and at least some overlap with the real nucleon. In this work we will use the following operator:

$$B_\alpha(t,\vec{p}) = \sum_{\vec{x}} e^{-i\vec{p}\vec{x}} \epsilon^{abc} u^a_\alpha(\vec{x},t) \left[u^b(\vec{x},t)^T C\gamma_5 d^c(\vec{x},t)\right] \tag{67}$$

We also use degenerate quark masses leading to degenerate nucleons.
In order to understand how we can extract information about the nucleon matrix elements using this operator we study two and three point correlation functions.

2.1.1 Two point correlation function

The two point correlation function is defined as

$$C_\Gamma(t,\vec{p}) = \Gamma_{\beta\alpha} \left\langle B_\alpha(t,\vec{p}) \bar{B}_\beta(0,\vec{p}) \right\rangle . \tag{68}$$

with an so far arbitrary Dirac matrix Γ. In order to project out states with positive parity we use the positive parity projector P_+ as defined in eq. (28)

$$\Gamma = P_+ = \frac{1+\gamma_4}{2}. \tag{69}$$

Using the transfer matrix formalism - an Euclidean analog of the application of the time evolution operator e^{-iHt} - on the two point correlation function leads to

$$C_\Gamma(t,\vec{p}) = \frac{\text{tr}\left[\Gamma S^{L_t-t} B(0,\vec{p}) S^t \bar{B}(0,\vec{p})\right]}{\text{tr}\, S^{L_t}}, \tag{70}$$

where $S = e^{-H}$ and L_t denotes the temporal extend.
The insertion of energy states identities[4] $\mathbb{1} = \sum_\nu |\nu\rangle\langle\nu|$ with $S|\nu\rangle = e^{-E_\nu}|\nu\rangle$ and the assumptions

$$L_t \to \infty,\ 0 \ll t \ll L_t \tag{71}$$

yields

$$C_\Gamma(t,\vec{p}) \approx \sum_{\vec{s}} \langle N_{\vec{s},\vec{p}}|\bar{B}(0,\vec{p})|0\rangle \Gamma \langle 0|B(0,\vec{p})|N_{\vec{s},\vec{p}}\rangle e^{-tE_\text{N}}. \tag{72}$$

The nucleon state dominates, as all states of higher mass are exponentially suppressed. The two almost degenerate, lowest negative parity states[5] get suppressed

[4] The states are given in lattice normalization $\langle\nu|\nu\rangle = 1$
[5] N(1535) and N(1650)

due to the Γ-contraction.
Eq. (72) can be further evaluated with the definition of the overlap

$$\langle 0|B_\alpha(0,\vec{p})|N_{\vec{s},\vec{p}}\rangle = Z u_\alpha(\vec{s},\vec{p}) \tag{73}$$

leading to[6]

$$C_\Gamma(t,\vec{p}) \approx |Z|^2 \sum_{\vec{s}} \bar{u}(\vec{s},\vec{p})\Gamma u(\vec{s},\vec{p}) e^{-tE_\mathrm{N}} = 2|Z|^2 \left(E_{\vec{p}} + m\right) e^{-tE_\mathrm{N}}. \tag{74}$$

The calculation of this correlator is also used to derive masses from the lattice. In the region where the assumptions (71) apply the mass can be extracted by a linear fit of the logarithm of this function.

2.1.2 Three point correlation function

The three point correlation function is defined as

$$C_\Gamma^{\mathcal{O}}(t,\tau,\vec{p}) = \Gamma_{\beta\alpha} \left\langle B_\alpha(t,\vec{p})\mathcal{O}(\tau)\bar{B}_\beta(0,\vec{p})\right\rangle, \tag{75}$$

where $\mathcal{O}(\tau)$ is the zero momentum superposition from eq. (62).
The right hand side of (75) can be reformulated in the transfer matrix formalism as

$$C_\Gamma^{\mathcal{O}}(t,\tau,\vec{p}) = \frac{\mathrm{tr}\left[\Gamma S^{L_t-t} B(0,\vec{p}) S^{t-\tau}\mathcal{O}(0) S^\tau \bar{B}(0,\vec{p})\right]}{\mathrm{tr}\; S^{L_t}}. \tag{76}$$

As we want to measure polarized matrix elements, Γ consists in addition to the parity projector also of spin projection operators (defined in eq. 29):

$$\Gamma_{\vec{s}} = P_+ P(\vec{s}) = \frac{1 \pm \gamma_4}{2} \frac{1 + i\gamma_5 \not{s}/m}{2} \tag{77}$$

Inserting eigenstates and using the approximations

$$L_t \to \infty,\; 0 \ll \tau \ll t \ll L_t \tag{78}$$

on eq. (76) yields

$$C_{\Gamma_{\vec{s}}}^{\mathcal{O}}(t,\tau,\vec{p}) \approx \sum_{\vec{\sigma}} \langle N_{\vec{\sigma},\vec{p}}|\mathcal{O}(0)|N_{\vec{\sigma},\vec{p}}\rangle \langle N_{\vec{\sigma},\vec{p}}|\bar{B}(0,\vec{p})|0\rangle \Gamma_{\vec{s}} \langle 0|B(0,\vec{p})|N_{\vec{\sigma},\vec{p}}\rangle e^{-tE_\mathrm{N}}, \tag{79}$$

where again all states with higher energy or negative parity do not contribute.
The insertion of overlap definitions as in eq. (73) leads to[7]

$$\begin{aligned} C_{\Gamma_{\vec{s}}}^{\mathcal{O}}(t,\tau,\vec{p}) \approx & |Z|^2 \sum_{\vec{\sigma}} \langle N_{\vec{\sigma},\vec{p}}|\mathcal{O}(0)|N_{\vec{\sigma},\vec{p}}\rangle \bar{u}(\vec{\sigma},\vec{p})\Gamma_{\vec{s}} u(\vec{\sigma},\vec{p}) e^{-tE_\mathrm{N}} \\ = & |Z|^2 \langle N_{\vec{s},\vec{p}}|\mathcal{O}(0)|N_{\vec{s},\vec{p}}\rangle (E_{\vec{p}} + m) e^{-tE_\mathrm{N}} \end{aligned} \tag{80}$$

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[6] $\sum_{\vec{s}} \bar{u}(\vec{s},\vec{p})\Gamma u(\vec{s},\vec{p}) = \sum_{\vec{s}} \mathrm{tr}[\Gamma u(\vec{s},\vec{p})\bar{u}(\vec{s},\vec{p})] \overset{(31)}{=} \sum_{\vec{s}} \mathrm{tr}\left[\Gamma\left(-i\not{p} + m\right) P(\vec{s})\right] \overset{(30)}{=}$
$\mathrm{tr}\left[\frac{1+\gamma_4}{2}\left(-i\not{p} + m\right)\right] = 2(E_{\vec{p}} + m)$

[7] $\bar{u}(\vec{\sigma},\vec{p})\Gamma_{\vec{s}} u(\vec{\sigma},\vec{p}) = \mathrm{tr}\left[P_+ P(\vec{s})\left(-i\not{p} + m\right) P(\vec{\sigma})\right] = \delta_{\vec{s},\vec{\sigma}}\, \mathrm{tr}\left[\frac{1+\gamma_4}{2}\frac{1 + i\gamma_5\not{s}/m}{2}\left(-i\not{p} + m\right)\right]$
$= \delta_{\vec{s},\vec{\sigma}}\left(E_{\vec{p}} + m\right)$

2.1.3 Ratios of correlation functions

Eq. (80) still contains the unknown overlap factor Z. But if we consider the ratio of the three and two point correlation function we find that its value equals half of the desired matrix element:

$$R(t,\tau,\vec{p},\vec{s},\mathcal{O}) = \frac{C^{\mathcal{O}}_{\Gamma_{\vec{s}}}(t,\tau,\vec{p})}{C_{\Gamma}(t,\vec{p})} \approx \frac{1}{2}\langle N_{\vec{s},\vec{p}}|\mathcal{O}|N_{\vec{s},\vec{p}}\rangle \approx \frac{1}{4E_{\vec{p}}}\langle N_{\vec{s},\vec{p}}|\mathcal{O}|N_{\vec{s},\vec{p}}\rangle_{\text{CONT}} \tag{81}$$

In the last step we changed from lattice (cp. footnote 4 on page 12) to continuum normalization (eq. 64).
This ratio seems not to depend on the arguments t and τ. This is of course only valid if the assumptions

$$L_t \to \infty,\ 0 \ll \tau \ll t \ll L_t \tag{82}$$

hold. Thus, for the functional dependence of R on τ for fixed t, one expects plateau-like behavior.

The calculations of this work have been performed with zero momentum and the antisymmetric spin combination

$$R_{\text{pol}}(t,\tau,\vec{0},\mathcal{O}) = R(t,\tau,\vec{0},\vec{s},\mathcal{O}) - R(t,\tau,\vec{0},-\vec{s},\mathcal{O}),\ \text{with } \vec{s}=(0,m,0), \tag{83}$$

which can be obtained by using

$$\Gamma_{\text{pol}} = \Gamma_{\vec{s}} - \Gamma_{-\vec{s}} = \frac{1+\gamma_4}{2} i\gamma_5\gamma_2 \tag{84}$$

in the three point correlation function.
To obtain the tensor charge we use one component of the operator from eq. (62) with $n = 2$. In this case the trace is zero and the symmetrization of only the second component is trivial.
As mentioned earlier, we calculate the non singlet contribution

$$\mathcal{O}^{24}_{\sigma,u-d} = \mathcal{O}^{24}_{\sigma u} - \mathcal{O}^{24}_{\sigma d} \tag{85}$$

and obtain

$$R_{\text{pol}}\left(t,\tau,\vec{0},\mathcal{O}^{24}_{\sigma,u-d}\right) \approx \frac{\left\langle N_{\vec{s},\vec{0}}\left|\mathcal{O}^{24}_{\sigma,u-d}\right|N_{\vec{s},\vec{0}}\right\rangle - \left\langle N_{-\vec{s},\vec{0}}\left|\mathcal{O}^{24}_{\sigma,u-d}\right|N_{-\vec{s},\vec{0}}\right\rangle}{4m}. \tag{86}$$

With use of eq. (63) this becomes

$$R_{\text{pol}}\left(t,\tau,\vec{0},\mathcal{O}^{24}_{\sigma,u-d}\right) \approx t_1^{(u-d)}. \tag{87}$$

2.2 Calculation of the correlators

Now that we know, what value the ratio of correlation functions would have, we need to calculate it on the lattice.

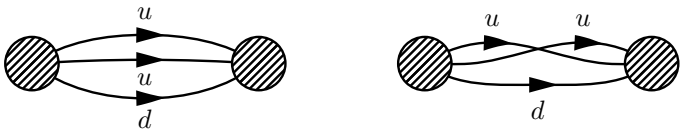

Fig. 1: The two graphs corresponding to the two contractions that contribute to eq. (88)

2.2.1 Two point correlation function

Therefore, we further evaluate the two point correlation function (eq. 68) using eq. (59) as shown in [1]. This leads to

$$C_\Gamma(t,\vec{p}) = \frac{-N_\mathrm{s}^3}{n}\sum_{n_i}\sum_{\vec{x}} e^{-i\vec{x}\vec{p}}\epsilon^{abc}\epsilon^{a'b'c'}\Big\{\mathrm{tr}\Big[\Gamma\, G_u^{aa'}(x,0)\Big]\times \\ \mathrm{tr}\Big[G_d^{cc'}(x,0)\,\tilde{G}_u^{bb'}(x,0)\Big] + \mathrm{tr}\Big[\Gamma\, G_u^{cc'}(x,0)\,\tilde{G}_d^{bb'}(x,0)\,G_u^{aa'}(x,0)\Big]\Big\}, \quad (88)$$

where the first sum is to be performed over the sample of gauge fields $\{A_i\}_{1\le i\le n}$, $x = (\vec{x}, t)$, $\tilde{G} = \gamma_5 C^{-1} G^{T_\mathrm{D}} C\gamma_5$ and $G^{ab}_{q,\alpha\beta}(c,x)$ is the inverse of the Dirac operator, i.e. the propagator, for quark flavor q, with color indices a, b and Dirac indices α, β. These propagators are functionals in the gauge fields $G\,[A_i]$.
In this form, we can actually calculate the correlator. First, a sample of gauge field configurations has to be generated. Next, the propagators for every gauge configuration have to be generated by inversion of the Dirac operator. And finally, the sum from eq. (88) can be performed.
In the derivation of eq. (88) one also uses the translational invariance to shift the sources of all propagators to zero. Of course, this invariance is only established in the full ensemble average of gauge configurations and thus just approximatively valid if we approximate the path integral. On the other hand, the saving of computation time is enormous, as the evaluation just a few rows of the propagator is cheap compared to the full inversion of the Dirac operator which has $V \cdot N_\text{color} \cdot N_\text{Dirac} \approx 6 \cdot 10^6$ rows.
Figure 1 shows the two possible contractions that contribute to eq. (88).

2.2.2 Three point correlation function

Before we head on to the three point correlation functions, it is convenient to rewrite the general operator $\mathcal{O}_q(\tau)$ as

$$\mathcal{O}_q(\tau) = \sum_{\vec{y},v,w} \bar{q}^a_\alpha(v)\mathcal{O}^{ab}_{\alpha\beta}(v,w;y)q^b_\beta(w), \quad \text{with } y = (\vec{y},\tau). \quad (89)$$

With this definition, the three point correlator (eq. 75) takes the following form (as shown in [1]):

$$C_\Gamma^{\mathcal{O}^{(u-d)}}(t,\tau,\vec{p}) = \frac{-N_\mathrm{s}^3}{n}\sum_{n_i}\sum_{\vec{y},v,w}\mathrm{tr}\Big[\Sigma_u^\Gamma(\vec{p},t;0,v)\;\mathcal{O}(v,w;y)\;G_u(w,0) - \Sigma_d^\Gamma(\vec{p},t;0,v)\;\mathcal{O}(v,w;y)\;G_d(w,0)\Big], \quad (90)$$

where the first sum is to be performed over the sample of gauge fields $\{A_i\}_{1\le i\le n}$.
Σ_q^Γ is called *sequential propagator*. It is defined as

$$\Sigma_q^\Gamma(\vec{p},t;0,v) = \sum_{\vec{x}} S_q^\Gamma(\vec{p};0,x)\;G_q(x,v) \quad (91)$$

S_q^Γ is called *sequential source*. The u-quark version is defined as

$$S_u^\Gamma(\vec{p};0,x) = e^{-i\vec{x}\vec{p}}\epsilon^{abc}\epsilon^{a'b'c'}\Big(\mathrm{tr}\Big[G_u^{bb'}(x,0)\;\tilde{G}_d^{cc'}(x,0)\Big]\;\Gamma + \tilde{G}_d^{cc'}(x,0)\;G_u^{bb'}(x,0)\Gamma + \Gamma G_u^{bb'}(x,0)\;\tilde{G}_d^{cc'}(x,0)\Big) \quad (92)$$

The graphs of the contractions corresponding to the three terms are shown in Figures 1(a), (c), (d) (in order of summation).
The d-quark version is defined as

$$S_u^\Gamma(\vec{p};0,x) = e^{-i\vec{x}\vec{p}}\epsilon^{abc}\epsilon^{a'b'c'}\,\mathrm{tr}\Big[\tilde{G}_u^{bb'}(x,0)\tilde{\Gamma}\tilde{G}_u^{cc'}(x,0)\Big] \quad (93)$$

The graphs of the contractions corresponding to this term is shown in Figures 1(h).

The $u-d$ subtraction cancels the remaining terms, because we use degenerate quarks. Figure 1(b) equals (g) because of the symmetric structure of the singlet part of our baryon operator (eq. 67). But most importantly figures 1(e) and (i), as well as figures 1(f) and (j), cancel. It is the disappearance of these *disconnected quark line* contributions which enabled us to write eq. (90) in the special form of

$$\mathrm{tr}[\Sigma(0,v)\mathcal{O}(v,w)G(w,0)]\,. \quad (94)$$

The process described by this term can be paraphrased in the following way: Propagation from the source to the operator with $G(w,0)$ (corr. to dashed lines in figure 2), insertion of the operator $\mathcal{O}(v,w)$ (corr. to the cross figure 2) and propagation back to the source with $\Sigma(0,v)$ (corr. to solid lines in figure 2).
$\Sigma(0,v)$ includes the propagation from the operator to the sink $G(x,v)$ and the propagation of two quarks from the source to the sink $S(x,0)$.
The benefits arising from the special form of the sequential propagator are not limited to a descriptive explanation of eq. (90). Instead eq. (91) can be rewritten as

$$\sum_v K_q(w,v)\;\gamma_5\Sigma_q^\Gamma(\vec{p},t;0,v)^\dagger = \gamma_5 S_q^\Gamma(\vec{p};0,w)^\dagger\;\delta_{w_0 t}, \quad (95)$$

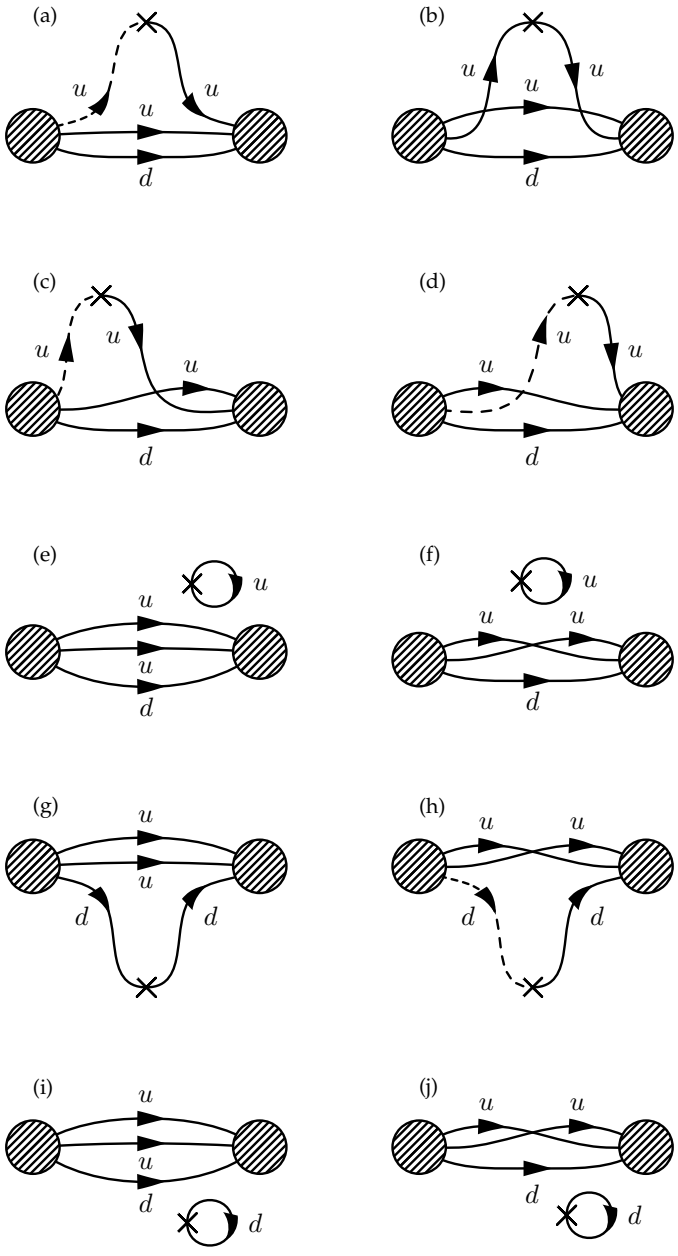

Fig. 2: All possible contractions from u-quark insertion ((a) through (f)) and d-quark insertion ((g) through (j)). The dashed lines correspond to the $G(w,0)$ part of eq. (90), the solid lines to the sequential propagators $\Sigma(0,v)$ and the cross to the operator insertion

where $K_q(w, v)$ is the Dirac operator of quark q. This equation implies that there is a very simple way of calculating the sequential propagator, namely the inversion of the Dirac operator. The computational advantage that arises hereby becomes evident if look at eq. (91): The straight forward calculation of the sequential propagator requires the propagators between any two point in our lattice, i.e. the full inversion of the Dirac operator, whereas the solution of eq. (95) is significantly less computationally intense.
So, we are now able to calculate the three point correlation function. The calculation involves the preparation of the gauge configuration sample and the determination of propagators $G(x, 0)$ - which has been performed already for the two point correlators. Then, the sequential propagators can be calculated as Greens functions from the sequential sources. And finally, the sum in eq. (90) has to be performed.

3 Calculation and results

For the calculations of this work I used gauge configurations, propagators and sequential propagators of the Bern-Graz-Regensburg Collaboration. The lattice had the dimensions

$$N_s^3 \times N_t = 16^3 \times 32. \tag{96}$$

I used a quenched ensemble of 100 gauge configurations that were generated using the Lüscher-Weisz action (see section 1.3) with

$$\beta = 7.90 \tag{97}$$

The scale as given by the lattice constant had been determined to be

$$a = 0.148(2)\,\mathrm{fm} \tag{98}$$

The spacial lattice volume is

$$V_s = 2.37(3)^3\,\mathrm{fm}^3. \tag{99}$$

The gauge fields have been smeared with hypercubic-blocking [5]. The propagators have been obtained by inversion of the chirally improved Dirac operator (see section 1.2; the coefficients used can be found in Appendix D of [1]). For bigger overlap factors Z (eq. 73) all appearing source quarks have been Jacobi smeared, so that the extend of the source roughly equals the size of a proton (see section 3.10.1 in [1]). Parameters used for Jacobi smearing are

$$\kappa = 0.21,\ N = 18,\ r \approx 0.354\,\mathrm{fm}. \tag{100}$$

Propagators were calculated with the following bare quark masses

$$am_q = 0.02,\ 0.03,\ 0.04,\ 0.05,\ 0.06,\ 0.08,\ 0.10,\ 0.12,\ 0.16,\ 0.20 \tag{101}$$

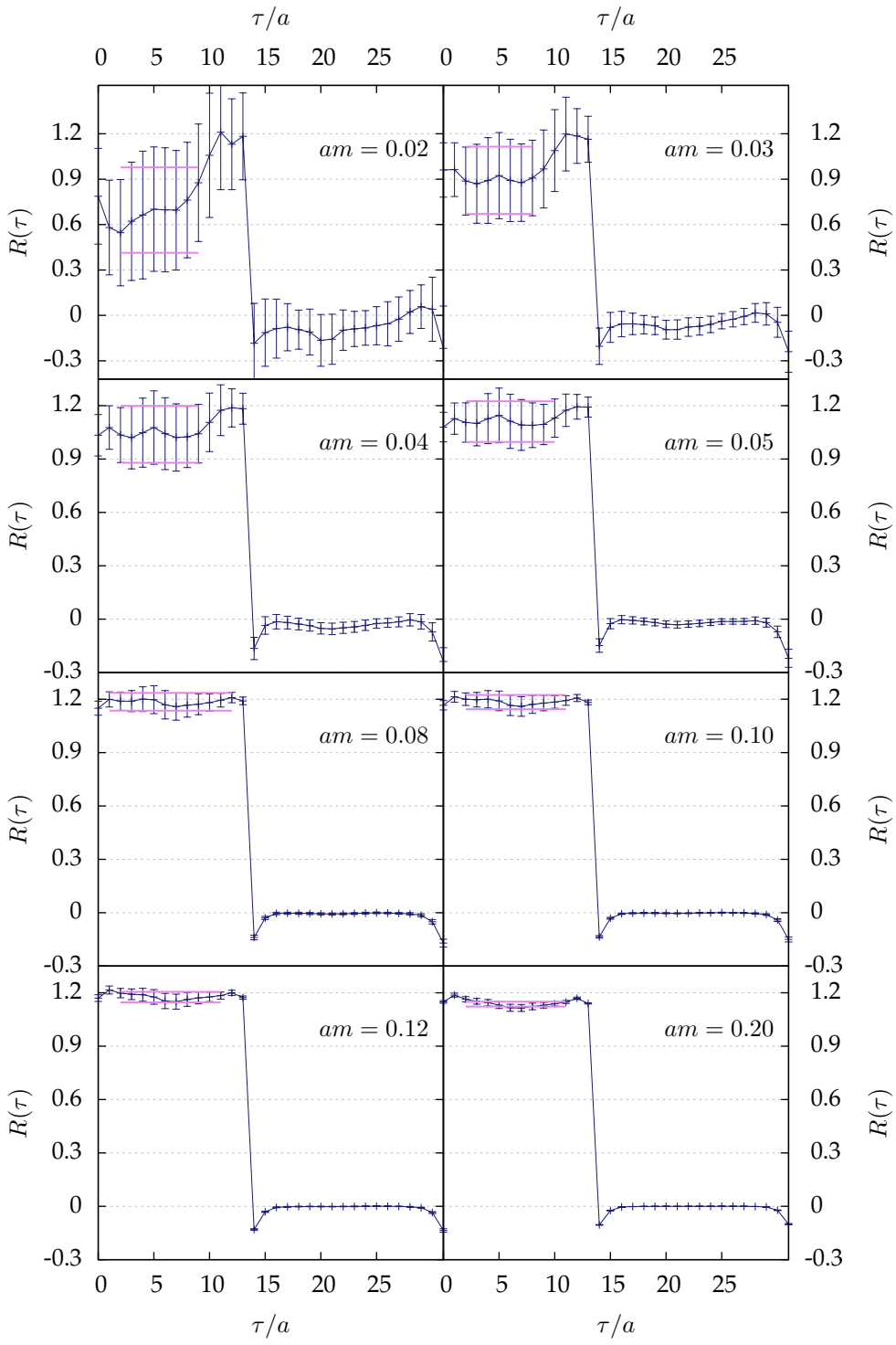

Fig. 3: $R_{\text{pol}}\left(t = 13a, \tau, \vec{0}, \mathcal{O}^{24}_{\sigma,u-d}\right)$ and the fitted plateau error bars

The nucleon and pion masses for this ensemble and some unpolarized matrix elements have been calculated by Thilo Maurer [1]. For the calculation of the tensor charge I used Thilo Maurer's FORTRAN code and extended it.

The result of my calculation is the ratio

$$R_{\text{pol}}\left(t = 13a, \tau, \vec{0}, \mathcal{O}^{24}_{\sigma,u-d}\right), \tag{102}$$

where $t = 13a$ has been chosen as a compromise between getting rid of higher energy states and minimizing the relative error. The values of the ratio and the results from the fitted plateau and the resulting error bars are shown in figure 3.

Table 1 summarizes the results and figure 4 contains corresponding plot.
R_t are the fitted plateau values of the ratio. The values of the tensor charge $t_1^{(u-d)}$ are given in the $\overline{MS}$ renormalization scheme at a scale of $\mu^2 = \mathrm{GeV}^2$. For the renormalization factor $t_1 = Z \cdot R_t$, I took the value from [1]:

$$Z_T^{\overline{MS}}(\mu^2) = 0.975(12) \tag{103}$$

As we only have data points from one ensemble with $a = 0.148\,\mathrm{fm}$, a continuum extrapolation is not possible. However, a chiral extrapolation to the physical or zero pion mass can be performed using these values.

am_q	m_π/MeV	R_t	$t_1^{\overline{MS}}$
0.02	344(8)	0.70(28)	0.68(28)
0.03	414(8)	0.89(22)	0.87(23)
0.04	473(9)	1.04(16)	1.01(17)
0.05	526(10)	1.11(11)	1.08(12)
0.08	659(11)	1.19(5)	1.16(6)
0.10	736(12)	1.18(4)	1.15(5)
0.12	807(13)	1.18(3)	1.15(4)
0.20	1050(16)	1.14(1)	1.11(3)

Table 1: The renormalized results of the tensor charge ($a = 0.148(2)fm$ and $Z_T^{\overline{MS}} = 0.975(12)$)

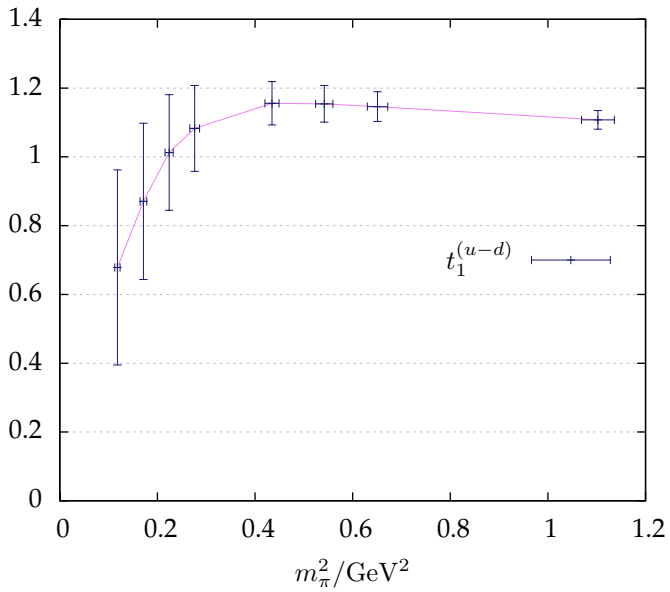

Fig. 4: Renormalized values of the tensor charge $t_1^{(u-d)}$ for different pion masses in the $\overline{MS}$ scheme at $\mu^2 = \mathrm{GeV}^2$

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